18/July/2014

Calasscal Mechanics

Newton's Laws of Rectilinear Motion :-

First Law :- {Relation of force and motion} :-

A law governing the relationship between forces and motion was completly described by Sir Issac Newton on the basis of inertia law of Galileo which asserted that if there are no external actions on a body it neither moves with a constant velocity or is at rest. Newton formulated this law as follows-

"A body will remain in its state of rest or uniform rectilinear motion (having zero acceleration) unless it is compelled to change its state by application of external force."

OR

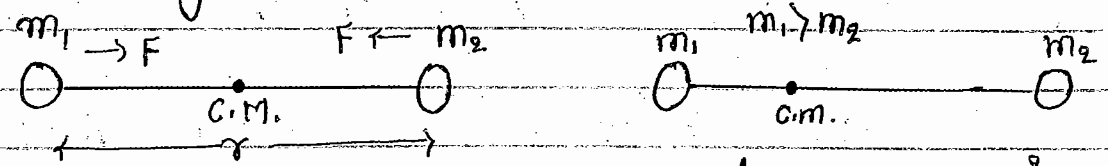
If the (vector) sum of all the forces acting on a given particle is zero then and only then the particle remains unaccelerated (i.e., remains at rest or moves with constant velocity).

If the sum of all the forces on a given particle is $\vec{F}$ and its acceleration is $\vec{a}$, the above statement may also be written as -

"$\vec{a} = 0$ if and only if $\vec{F} = 0$"

Thus, if the sum of the forces acting on a particle is known to be zero, we can be sure that the particle is unaccelerated, or if we know that a particle is unaccelerated, we can be sure that the sum of forces acting on the particle is zero.

* ~~Second Law :-~~ If no external force acts on a system, the state of system does not change.



{ For smooth surface }

{ $F_{external} = 0$ }
then C.m. is at rest

$$\boxed{F = \frac{G m_1 m_2}{r^2}}$$

Here,

State $\longrightarrow$ Centre of mass

If system consists of more than one particle then newton's law is collectively applied to centre of mass.

* Second Law of Motion :-

"The acceleration of a particle as measured from an inertial frame is given by the (vector) sum of all the forces acting on the particle divided by its mass."

In symbols,

$$\vec{a} = \frac{\vec{F}}{m}$$

$$\boxed{\vec{F} = m\vec{a}} \longrightarrow \circledast$$

A force $\vec{F}$ acting on a particle of mass m produces an acceleration $\vec{F}/m$ in it with respect to an inertial frame. This is law of nature.

If the forces ceases to act at some instant the acceleration becomes zero at the same instant. In equation ⊛ $\vec{a}$ and $\vec{F}$ are measured at the same instant of time.

OR

"The rate of change of linear momentum of a body is equal to the force acting on it."

Thus,

$$\vec{F}_{ext} = \frac{d(\vec{p})}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} \quad \left\{\text{taking mass constant}\right\} \quad \left\{\vec{p} = \text{linear momentum}\right\}$$

$$\boxed{\vec{F}_{ext} = m\vec{a}} \longrightarrow (1)$$

valid only if m = Constant

OR

$$\vec{F}_{ext} = m\frac{d^2\vec{r}}{dt^2} \Rightarrow \text{or} \quad \frac{d^2\vec{r}}{dt^2} = \frac{\vec{F}}{m} = \vec{a}$$

$$\text{or} \quad \boxed{\vec{F} = m\vec{a}}$$

If mass is variable then law is written as-

$$\boxed{\vec{F} = \frac{d(m\vec{v})}{dt} = \vec{v}\frac{dm}{dt} + m\frac{d\vec{v}}{dt}} \longrightarrow (2)$$

If more than one force acts on the body which consequently acquires acceleration $\vec{a}$, then we write-

$$\boxed{\Sigma\vec{F} = m\vec{a}} \longrightarrow (3)$$

Equation (3) is the mathematical form of Newton's Second law and provides for the general equation of rectilinear motion which is the basic equation of the classical mechanics.

Note :-

(i) Newton's first law is simply a special case of second law.

(ii) Both law hold if the frame of reference has no acceleration. {Inertial reference frame}.

(iii) A frame of reference in which Newton's first and second law hold is called an "Inertial" or "Gallilean" or "Newtonian" frame of reference.

* Newton's Third Law of Motion :-

Newton's third law states that "If a body 'A' exerts a force $\vec{F}$ on another body 'B', then 'B' exerts a force $-\vec{F}$ on 'A'.

OR

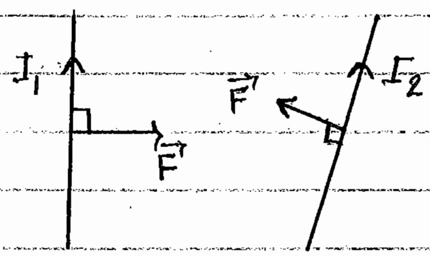
"To every action there is always an equal and opposite reaction."

Note :-

Forces always occur in pairs. Forces on a body A and B is equal & opposite to the force on the body B by A.

$\boxed{\vec{F}_{AB} = -\vec{F}_{21}}$ → always valid in mechanical cases.

Violation :-



In this case magnetic force between two parallel wires are not opposite in direction.

Note :-

(i) Newton's third law can be used in non-inertial reference frame, if psudo force are also included.

Resultant known force directly acting on system along X. i.e. $F_x = m a_x$.

(ii) A single isolated force is impossible.

(iii) One of the two forces, involved in mutual interaction between two bodies, is called "action force" and other is termed "reaction force."

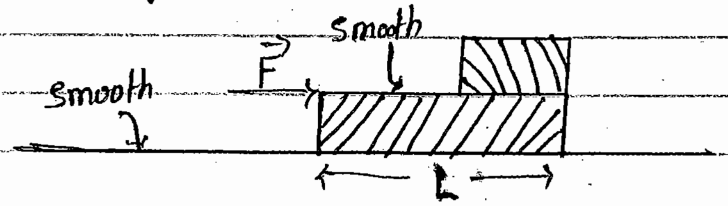
(iv) Third law speaks of mutual and simultaneous interaction.

(v) Action and reaction forces which always occur in pairs, act on different bodies.

Equation of motion $F_x = m\ddot{x}$

* Problem based on Newton's Law's :-

Ques From the given diagram, After what time the two block will separate?



Sol^n There small block will remain at rest & big block will pass from below

$\therefore$ Acceleration of big block:-

$$\vec{a} = \frac{F_{ext}}{mass} = \frac{\vec{F}}{M}$$

$\because$ $\vec{F}$ and M is constant so $\vec{a}$ is constant.

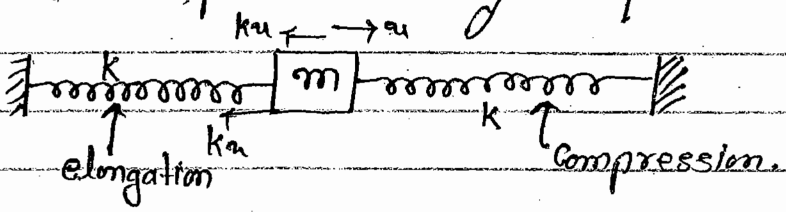
$\therefore$ $s = ut + \frac{1}{2}\vec{a}t^2$

$$L = 0 + \frac{1}{2}\frac{\vec{F}}{M}t^2$$

$$\boxed{t = \sqrt{\frac{2LM}{\vec{F}}}}$$

This is the time after which the two block separate.

A-1 Ques 12 In the fig. shown equation of motion of block which is displaced horizontally is-



Sol^n Direction of motion = +ve

Equation of motion,

$$\vec{F} = m\ddot{x}$$

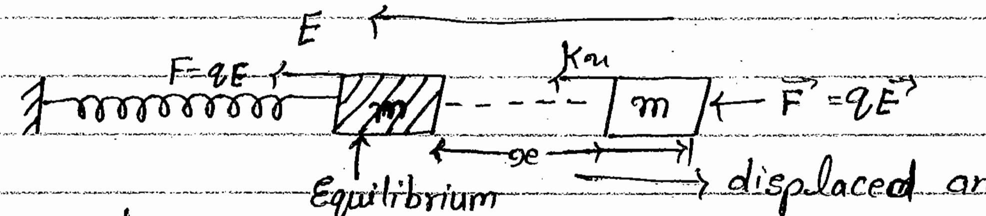
$$-kx - kx = m\ddot{x}$$

$$-2kx = m\ddot{x}$$

$$-\frac{2K}{m}x = \ddot{x}$$

So $\boxed{\ddot{x} + \frac{2K}{m}x = 0}$ Ans

Qus Write equation of motion for given figure if the block is displaced to the +ve x-direction & then



Sol^n Equation of motion :-

$$-kx - q\vec{E} = m\ddot{x}$$

$$-(kx + q\vec{E}) = m\ddot{x}$$

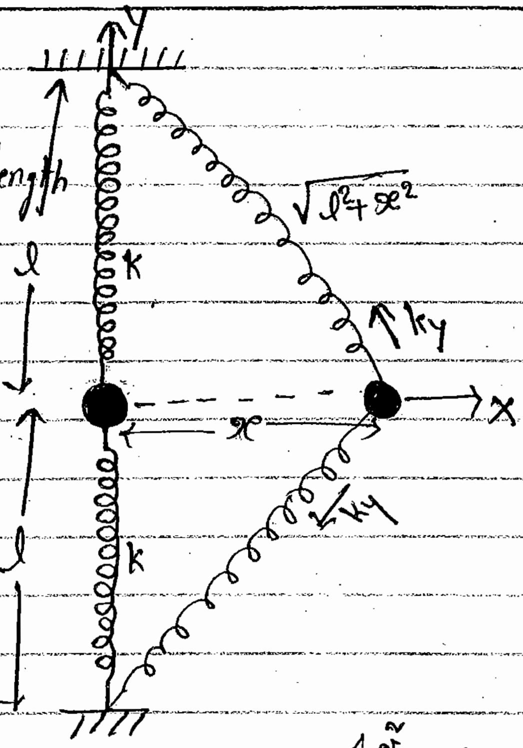
$$\boxed{m\ddot{x} = -(kx + q\vec{E})}$$ Ans

Qus Consider a particle of mass m attached to two identical springs each of length 'l' & spring constant K. The equilibrium configuration is the one where the springs are unstretched. There are no other external forces on the system. If the particle is given a small displacement along the x-axis, what will be the equation of motion for small oscillations?

Sol^n Let y is elongation

Elongation = Final length - initial length

$$y = \sqrt{x^2+l^2} - l$$



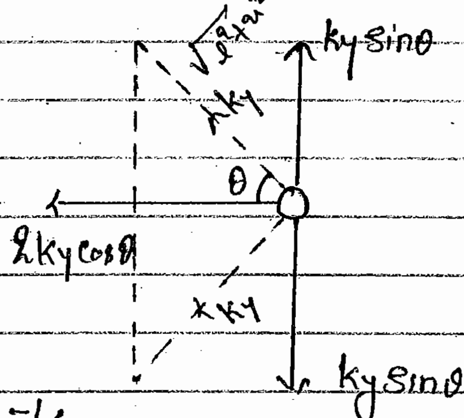
Equation of motion -

$$F = m\ddot{x}$$

$$-2ky\cos\theta = m\ddot{x}$$

$$-2k\left[\sqrt{x^2+l^2} - l\right]\frac{x}{\sqrt{l^2+x^2}} = m\ddot{x}$$

$$\Rightarrow -2kx\left[1 - \frac{l}{\sqrt{l^2+x^2}}\right] = m\ddot{x}$$



for small displacement -

$$x << l$$

use binomial expansion -

$$\Rightarrow \frac{l}{\sqrt{x^2+l^2}} = \frac{\cancel{l}}{\cancel{l}\sqrt{1+\frac{x^2}{l^2}}} = \left(1+\frac{x^2}{l^2}\right)^{-1/2}$$

$$= 1 - \frac{x^2}{2l^2} + \cdots$$

$$\therefore \quad -2kx\left[\cancel{1}-\cancel{1}+\frac{x^2}{2l^2}\right] = m\ddot{x}$$

$$\frac{-kx^3}{l^2} = m\ddot{x}$$

$$\Rightarrow \boxed{m\ddot{x} + \frac{kx^3}{l^2} = 0}$$

IInd Method:-

There is another method -

Since there is no friction. So energy is constant.

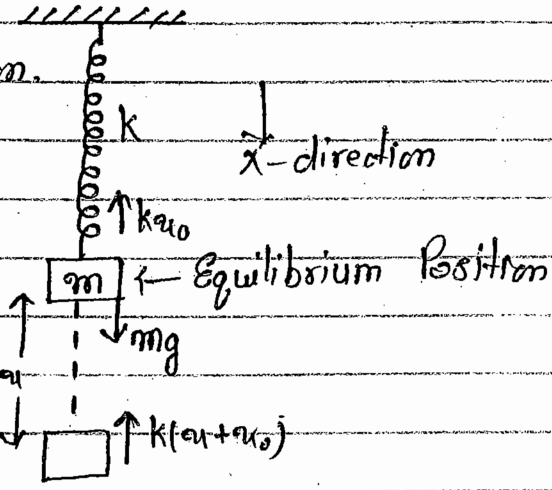
$$\text{Energy} = K.E. + P.E.$$
$$= \frac{1}{2} m (\dot{x})^2 + 2 \times \frac{1}{2} k [\text{elongation}]^2$$
$$= \frac{1}{2} m (\dot{x})^2 + \frac{2 . k}{2} [\sqrt{x^2 + l^2} - l]^2$$

Differentiate w.r. to time and use $\frac{dE}{dt} = 0$

Ques 4 In the fig. shown if the block is displaced from equilibrium position by a distance 'x' its equation of motion will be-

Sol^n Let x_0 is initial elongation.

For Equilibrium -



$$kx_0 = mg \quad \text{——} (1)$$

Equation of motion -

$$F_x = m\ddot{x}$$

$$mg - k(x+x_0) = m\ddot{x}$$

displacement = +ve

$$\cancel{mg} - kx - \cancel{kx_0} = m\ddot{x} \qquad \{\because \text{ at equilibrium } kx_0 = mg\}$$

$$-kx = m\ddot{x}$$

$$\boxed{\ddot{x} + \frac{k}{m}x = 0}$$

For Spring mass System:- If equation of motion is written in terms of displacement from equilibrium position then 'mg' does not cause in final equation.

* If in above question the elongation is y then equation of motion:-

$$\vec{F}_y = m\ddot{y} \Rightarrow mg - ky = m\ddot{y} \Rightarrow \ddot{y} = g - \frac{k}{m}y$$

$$\Rightarrow \boxed{\ddot{y} + \frac{ky}{m} - g = 0} \quad \underline{\text{Ans}}$$

* Kinematics :-

$$v_x = \frac{dx}{dt}$$

$$a_x = \frac{dv_x}{dt}$$

or $V = \frac{dx}{dt}$, $a_x = \frac{dV_x}{dx} \cdot \frac{dx}{dt} \rightarrow V_x$

$$a_x = v_x \frac{dv_x}{dx}$$

or $a = V \frac{dV}{dx}$

If a = Constant

Definations.

Valid only for $\vec{a} \neq$ Const.

Formulas :-

$$S = ut + \frac{1}{2}\vec{a}t^2$$

$$V = u + \vec{a}t$$

$$V^2 = u^2 + 2\vec{a}\vec{s}$$

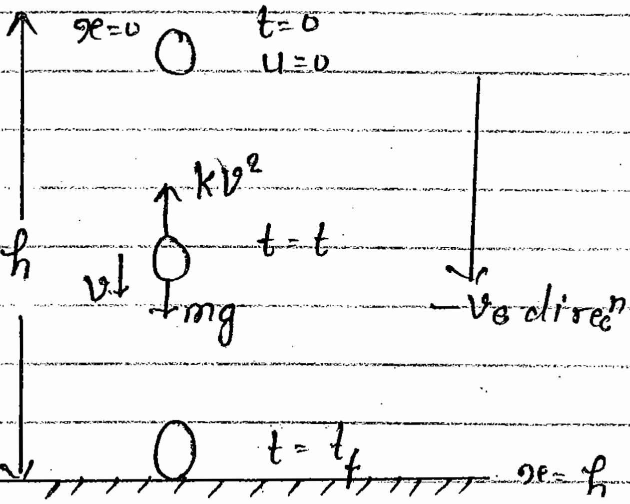
Valid only for $\vec{a}$ = Constant.

A-1

Ques 8 A body of mass 'm' falls from rest at a hight 'h' under gravity (acceleration due to gravity g) through a dense medium which provides a resistive force $F = -kv^2$, where 'k' is a constant & 'v' is the speed. It will hit the ground with what K.E. ?

Sol^n

Let v is speed at time t.



Equation of motion -

$$F_x = ma_x$$

$$mg - kv^2 = ma$$

$$a = g - \frac{k}{m}v^2$$

$\because$ Acceleration is variable

$$\therefore \quad v\frac{dv}{dx} = g - \frac{k}{m}v^2$$

$$\Rightarrow \int_0^{v_f} \frac{v\,dv}{\cancel{dx}(g - \frac{k}{m}v^2)} = \int_0^h dx$$

Let $g - \frac{k}{m}v^2 = y$

$-\frac{k}{m}2v\,dv = dy$

$v\,dv = -\frac{m}{2k}dy$

$$\Rightarrow \int_0^{v_f} \frac{-\frac{m}{2k}dy}{y} = h$$

$$-\frac{m}{2k}[\log y] = h$$

$$-\frac{m}{2k}\left[\log\left(g - \frac{kv^2}{m}\right)\right]_0^{v_f} = h$$

$$h = -\frac{m}{2k}\log\left\{\frac{g - \frac{kv_f^2}{m}}{g}\right\}$$

$$\log\left(1-\frac{kv_f^2}{mg}\right) = -\frac{2k}{m}h$$

$$1-\frac{kv_f^2}{mg} = e^{-\frac{2k}{m}h}$$

$$v_f^2 = \frac{mg}{k}\left(1-e^{-\frac{2kh}{m}}\right)$$

$$\boxed{K.E. = \frac{1}{2}mv_f^2 = \frac{m^2g}{2k}\left(1-e^{-\frac{2kh}{m}}\right)}$$

Ans

Trick to test options :-

If here k tends to zero, i.e. $k\to 0$ then this should give $K.E. = mgh$.

Take limit $k \to 0$

$$K.E. = \frac{m^2g}{2k}\left(1-e^{-\frac{2kh}{m}}\right)$$

Let $x = \frac{2kh}{m}$

$$e^{-x} = 1-x+\frac{x^2}{\underline{|2}} - \dots$$

$$\simeq 1-x \quad \{\because x\to 0\}$$

$$\therefore \quad K.E. = \frac{m^2g}{2k}\left(\cancel{1}-\cancel{1}+\frac{2kh}{m}\right)$$

$$\boxed{K.E. = mgh}$$

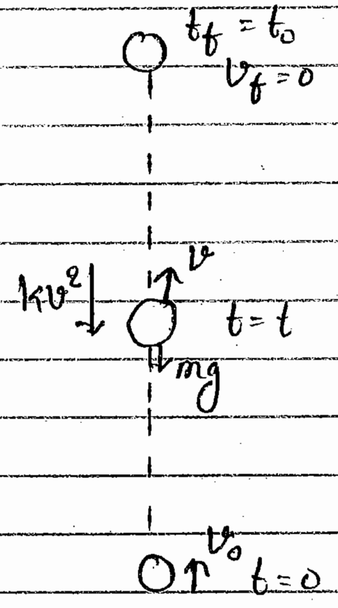
Ans

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

A-1

Qus. 9 A particle of mass m is thrown upward with velocity v & there is retarding air resistance proportional to the square of the velocity with proportionality constant k. If the particle attains the maximum hight after time t, and g is the gravitational acceleration what is the velocity v?

Sol^n: Equation of motion:-



$$-mg - kv^2 = ma$$

$$a = -\left(g + \frac{k}{m}v^2\right)$$

$$\frac{dv}{dt} = -\left(g + \frac{k}{m}v^2\right)$$

$$\int_{v_0}^{0} \frac{dv}{(\sqrt{g})^2 + \left(\sqrt{\frac{k}{m}}\,v\right)^2} = -\int_0^{t_0} dt$$

$$\frac{\sqrt{m}}{\sqrt{k}}\,\frac{1}{\sqrt{g}}\left[\tan^{-1}\frac{\sqrt{\frac{k}{m}}\,v}{\sqrt{g}}\right]_{v_0}^{0} = -t_0$$

$$\sqrt{\frac{m}{kg}} \times \left[\tan^{-1}0 - \tan^{-1}\sqrt{\frac{k}{mg}}\,v_0\right] = -t_0$$

$$\sqrt{\frac{m}{kg}}\tan^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right) = t_0$$

$$\tan^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right) = \sqrt{\frac{kg}{m}}\,t_0$$

$$\sqrt{\frac{k}{mg}}\,v_0 = \tan\left(\sqrt{\frac{kg}{m}}\,t_0\right)$$

$$V_0 = \sqrt{\frac{mg}{k}}\tan\left(\sqrt{\frac{kg}{m}}\,t_0\right)$$

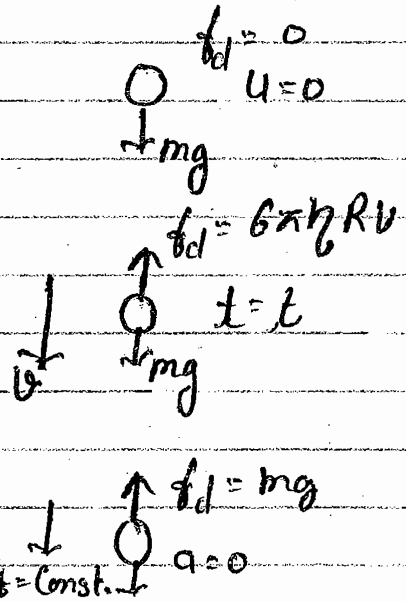
Put $m = 1$

$$\boxed{V_0 = \sqrt{\frac{g}{k}}\tan\left(\sqrt{kg}\,t_0\right)}$$

Net 2013 Dec.

Ques 6 A-1 Spherical particle of a given material of density ρ are released from rest inside a liquid medium of lower density. The viscous drag force may be approximated by the stoke's law i.e. $F_d = 6\pi\eta R v$, where η is the viscosity of the medium, R the radius of a particle & v its instantaneous velocity. If $\tau(m)$ is the time taken by a particle of mass m to reach half its terminal velocity, then the ratio $\tau(8m)/\tau(m)$ is - ?

Sol^n When terminal velocity is reached -



$$F_d = mg$$

$$\text{density} = \frac{\text{mass}}{\text{Volume}}$$

$$\rho = \frac{m}{\frac{4}{3}\pi R^3}$$

$$\cancel{R^3 = 6\pi\eta R v = 6\pi\eta\left[\frac{3m}{4\pi\rho}\right]^{1/3} v}$$

$$R^3 = \frac{3m}{4\pi\rho} \Rightarrow R = \left[\frac{3m}{4\pi\rho}\right]^{1/3}$$

$$F_d = 6\pi\eta R v = 6\pi\eta\left[\frac{3m}{4\pi\rho}\right]^{1/3} v$$

$$\boxed{F_d = k m^{1/3} v} \quad \text{where } k = 6\pi\eta\left[\frac{3}{4\pi\rho}\right]^{1/3}$$

Now

$$F_d = mg \Rightarrow k m^{1/3} v_t = mg \Rightarrow \boxed{v_t = \frac{m^{2/3}}{k} g} \quad (1)$$

here $v_t \Rightarrow$ terminal velocity

Now,

Equation of motion (to find acc^n)

$$mg - F_d = ma$$

$$mg - k m^{1/3} v = ma$$

$$a = g - k m^{-2/3} v$$

$$\frac{dv}{dt} = g - k m^{-2/3} v$$

$$\int_0^{v_t/2} \frac{dv}{g - k m^{-2/3} v} = \int_0^{\tau} dt$$

$$\left[\frac{-1}{k m^{-2/3}} \log\left(g - k m^{-2/3} v\right)\right]_0^{\frac{m^{2/3} g}{2k}} = \tau$$

$$\Rightarrow \frac{-m^{2/3}}{k}\left[\log\left(g - \frac{g}{2}\right) - \log g\right] = \tau$$

$$\Rightarrow \frac{-m^{2/3}}{k} \log \frac{g/2}{g} = \tau$$

$$\Rightarrow \frac{-m^{2/3}}{k} \log\left(\frac{1}{2}\right) = \tau$$

$$\Rightarrow \boxed{\frac{m^{2/3}}{k} \log 2 = \tau}$$

Now

$$\tau(m) = \frac{m^{2/3}}{k} \log 2$$

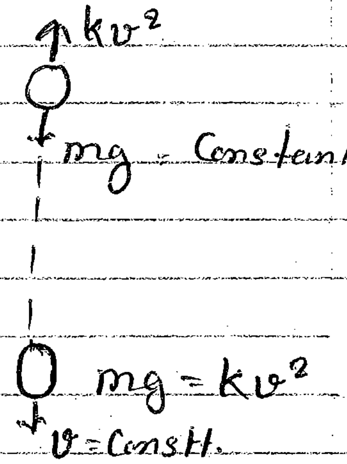
$$\tau(8m) = \frac{(8m)^{2/3}}{k} \log 2$$

$$\boxed{\frac{\tau(8m)}{\tau(m)} = 8^{2/3} = 4}$$

Ans

A-1
Ques 7. A particle is observed after it has been moving for a long time under the influence of a constant force in a medium that applies a drag force proportional to the square of its velocity. Distance versus time graph made by the observer will look like?

Solⁿ for constant velocity, distance v/s time graph will be straight line.



{∵ Particle is moving for a long time. So after reaching to the terminal velocity, the velocity becomes constant.}

∴ Terminal velocity, $v = \frac{dx}{dt}$

$$dx = v\,dt$$

∵ v is constant.

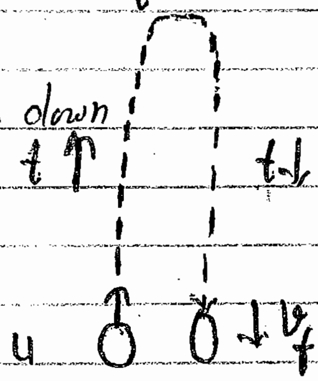
$$\therefore \int dx = v \int dt$$

$$\boxed{x = vt + c}$$ ← It is eqⁿ of straight line.

So it is straight line.

* Vertical motion in absence of air resistance

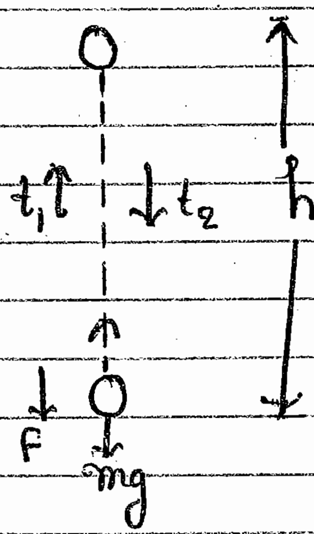
time for going up = time for going down



$$\boxed{\begin{aligned} t\uparrow &= t\downarrow \\ u\uparrow &= v_f\downarrow \end{aligned}}$$

A-2
Ques. An object of mass m is thrown vertically up. It is acted upon by a constant resistive force F. If t_1 & t_2 be time of ascent & time of descent then value of t_1/t_2 is ?

Sol^n:- Let h be the maximum hight reached.



Equation of motion -

$$-mg - F = ma$$

$$\boxed{a = -\left(g + \frac{F}{m}\right)} = \text{Constant}$$

$$v = u + at_1$$

$$0 = u - \left(g + \frac{F}{m}\right)t_1$$

$$\boxed{t_1 = \frac{u}{\left(g + \frac{F}{m}\right)}} \longleftarrow (1)$$

Express u in terms of h :-

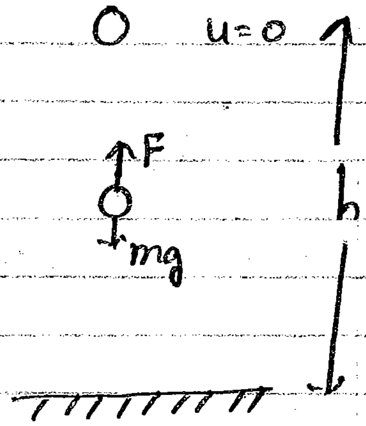
$$v^2 = u^2 + 2as$$

$$\Rightarrow \quad 0 = u^2 - 2\left(g + \frac{F}{m}\right) \cdot h$$

$$u = \sqrt{2\left(g + \frac{F}{m}\right)h}$$

$$\therefore \quad \boxed{t_1 = \frac{\sqrt{2h}}{\sqrt{g + \frac{F}{m}}}} \quad \text{—— (ii)}$$

for downward motion :-



Equation of motion :-

$$mg - F = ma$$

$$\boxed{a = g - \frac{F}{m}} = \text{constant.}$$

$$s = ut + \frac{1}{2}at^2$$

$$h = 0.t + \frac{1}{2}\left(g - \frac{F}{m}\right)t_2^2$$

$$h = 0 + \frac{1}{2}\left(g - \frac{F}{m}\right)t_2^2$$

$$t_2 = \sqrt{\frac{2h}{(g - F/m)}}$$

$$\boxed{\frac{t_1}{t_2} = \sqrt{\frac{g - F/m}{g + F/m}}} \quad (t_1 < t_2)$$

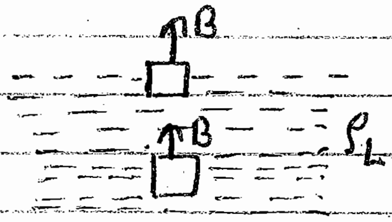
Ans

mass = Volume x density

* Buoyancy Force (B) :-

B = weight of displaced

B = mass of liquid displaced x g.



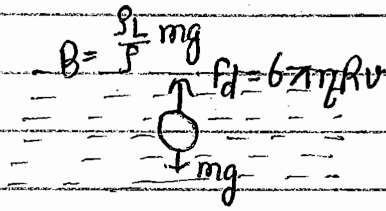
B = Volume of liquid displaced . ρ_L . g.

$$\boxed{B = \text{Volume of object inside liquid} \cdot \rho_L \cdot g}$$

ρ_L = density of liquid.

A-1
Q. 6 { Icluding Bouyancy force }

$f_d = 6\pi\eta R v = k m^{1/3} v$



$B = \text{Vol.}\, \rho_L \cdot g$

When v_t is reached

$B + f_d = mg$

$\text{Vol.}\, \rho_L \cdot g + k m^{1/3} v = mg$

$\underbrace{\text{Vol.}\, \rho}\, \frac{\rho_L}{\rho}\, g + k m^{1/3} v_t = mg$

$m \frac{\rho_L}{\rho} g + k m^{1/3} v_t = mg$

$k m^{1/3} v_t = mg \left(1 - \frac{\rho_L}{\rho}\right)$

Equation of motion -

$$mg - \frac{\rho_L}{\rho} mg - k v m^{1/3} = ma$$

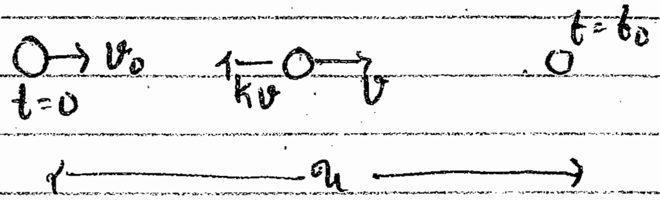
$$\boxed{g\left(1 - \frac{\rho_L}{\rho}\right) - K m^{-2/3} v = a}$$

A-2

1.4 A particle of mass m is thrown with initial speed v_0. A resistance force $-kv$ acts on the particle. Distance moved by particle in time t_0?

Sol^n

Equation of motion -



$$-kv = ma$$

$$-kv = m\frac{dv}{dt}$$

$$-\frac{k}{m}\int_0^t dt = \int_{v_0}^{v}\frac{dv}{v} \Rightarrow \frac{-k}{m}t = \log\frac{v}{v_0}$$

$$\Rightarrow v = v_0 e^{-k/m\, t}$$

$$\Rightarrow \frac{dx}{dt} = v_0 e^{-\frac{k}{m}t}$$

$$\Rightarrow \int_0^x dx = v_0\int_0^{t_0} e^{-\frac{k}{m}t}\,dt$$

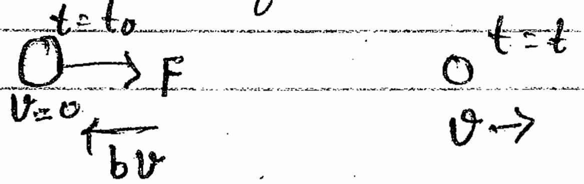
$$x = \frac{v_0}{-k/m}\left[e^{-\frac{k}{m}t}\right]_0^{t_0}$$

$$\boxed{x = \frac{m v_0}{k}\left(1 - e^{-\frac{k}{m}t_0}\right)} \qquad \{e^{-0} = 1\}$$

A-2

Ques 2 A constant force F is applied on a particle of mass 'm' which is initially at rest. As the particle starts moving a resistive force $-bv$ begins to act on it. Speed of the particle at any instant of time t is?

Sol^n



Equation of motion -

$$-bv + F = ma$$

$$\boxed{a = \frac{-bv+F}{m}} \qquad \left\{ \begin{array}{l} \text{accelaration} \\ \text{is variable} \end{array} \right\}$$

$$\frac{dv}{dt} = \frac{-bv+F}{m}$$

$$\Rightarrow m\int_0^v \frac{dv}{(-bv+F)} = \int_0^t dt$$

$$\Rightarrow m \times \frac{1}{-b}\left[\log(-bv+F)\right]_0^v = t$$

$$\Rightarrow \log(-bv+F) - \log F = -\frac{bt}{m}$$

$$\Rightarrow \log\frac{(-bv+F)}{F} = \frac{-bt}{m}$$

$$\Rightarrow \frac{-bv+F}{F} = e^{-bt/m}$$

$$\Rightarrow \frac{\cancel{+}bv}{F} = \cancel{+}(1-e^{-bt/m})$$

$$\Rightarrow \boxed{v = \frac{F}{b}(1-e^{-bt/m})}$$

Ans

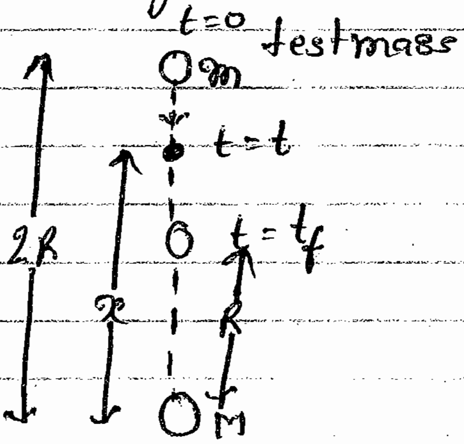
A-1

Q.10 Find the free fall time of a test mass on an object of mass M from a hight 2R to R.

Sol^n Let at time t distance of test mass from the given object is x

Then eq^n of motion -

$$\frac{G\cancel{m}M}{x^2} = \cancel{m}a$$



$$2\sin^2\theta = 1 - \cos 2\theta$$

$$\frac{GM}{x^2} = -v\frac{dv}{dx}$$ { ∵ x is decreasing so $-ve$ sign is taken

$$\Rightarrow \int_{2R}^{R} \frac{GM}{x^2}\,dx = -\int_0^v v\,dv$$

$$\Rightarrow GM\left[\frac{1}{x} - \frac{1}{2R}\right] = \frac{v^2}{2}$$

$$\Rightarrow GM\left[\frac{2}{x} - \frac{1}{R}\right] = v^2$$

$$\sqrt{GM}\sqrt{\frac{2R-x}{Rx}} = v$$

Note:-

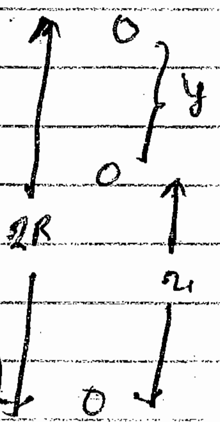
$$v = \frac{dy}{dt}$$

$$v = \frac{d}{dt}(2R-x)$$

$$v = -\frac{dx}{dt}$$

$$a = \frac{dv}{dt} = \frac{dv}{dx}\left(\frac{dx}{dt}\right)$$

$$\boxed{a = -v\frac{dv}{dx}}$$



$$\sqrt{\frac{GM}{R}}\sqrt{\frac{2R-x}{x}} = -\frac{dx}{dt}$$

$$\int_0^{t_f}\sqrt{\frac{GM}{R}}\,dt = -\int_{2R}^{R}\sqrt{\frac{x}{2R-x}}\,dx$$

Let $x = 2R\sin^2\theta$

$dx = 4R\sin\theta\cos\theta\,d\theta$

$\sin^2\theta = \frac{x}{2R}$

$\sin\theta = \sqrt{\frac{x}{2R}}$

$\theta = \sin^{-1}\sqrt{\frac{x}{2R}}$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$\cos\theta = \sqrt{1-\frac{x}{2R}}$

$$\Rightarrow \sqrt{\frac{GM}{R}}\,t_f = -\int_{2R}^{R}\sqrt{\frac{2R\sin^2\theta}{2R\cos^2\theta}}\,.\,4R\sin\theta\cos\theta\,d\theta$$

$$= -2R\int_{2R}^{R} 2\sin^2\theta\,d\theta$$

$$= -2R\int_{2R}^{R}[1-\cos 2\theta]\,d\theta$$

$$= -2R\left[\theta - \frac{\sin 2\theta}{2}\right]_{2R}^{R}$$

$$= -2R\left[\sin^{-1}\sqrt{\frac{x}{2R}} - \sqrt{\frac{x}{2R}}\sqrt{1-\frac{x}{2R}}\right]_{2R}^{R}$$

$$= \left[\frac{\pi}{4} - \frac{\pi}{2} - \frac{1}{2} + 0\right]$$

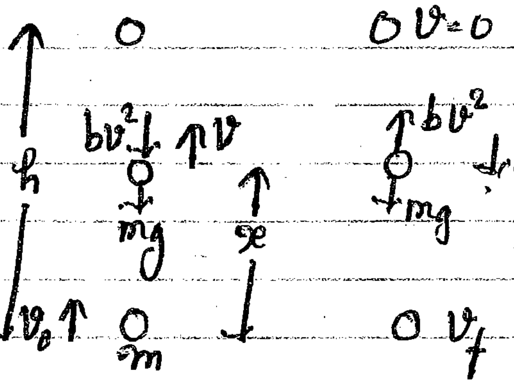
$$\sqrt{\frac{GM}{R}}\, t_f = R\left[\frac{\pi}{2} + 1\right]$$

$$\boxed{t_f = \left[\frac{\pi}{2} + 1\right]\sqrt{\frac{R^3}{GM}}}$$

A-2

Q.11 A particle of unit mass is thrown vertically upward with initial speed v_0. The particle is acted upon by a drag force bv^2 in addition to gravity. Here 'b' is a constant and v is instantaneous speed. Speed of the particle when it returns to the point from where it had been thrown is -

Sol^n For Upward motion :-



$$-mg - bv^2 = ma$$

$$-\left(g + \frac{bv^2}{m}\right) = v\frac{dv}{dx}$$

$$-\int_0^h dx = \int_{v_0}^{0} \frac{v\,dv}{\left(g + \frac{bv^2}{m}\right)}$$

$$h = -\frac{m}{2b}\left[\log\left[g + \frac{bv^2}{m}\right]\right]_{v_0}^{0}$$

$$\boxed{h = \frac{-m}{2b}\left[\log\left(\frac{g}{g + \frac{bv^2}{m}}\right)\right]} \quad \text{—} (1)$$

For downward motion :-

$$mg - bv^2 = ma \Rightarrow v\frac{dv}{dx} = g - \frac{bv^2}{m}$$

$$\Rightarrow \int_0^{v_f} \frac{v\,dv}{\left(g - \frac{bv^2}{m}\right)} = \int_0^{h} dx \Rightarrow \frac{-m}{2b}\left[\log \frac{\left(g - \frac{bv_f^2}{m}\right)}{g}\right] = h$$

$$\boxed{h = \frac{-m}{2b}\left[\log \frac{\left(g - \frac{bv_f^2}{m}\right)}{g}\right]} \quad \text{— (ii)}$$

from eq^n (i) and (ii), we get:-

$$\frac{-\cancel{m}}{\cancel{2b}}\left[\log \frac{g}{\left(g + \frac{bv_0^2}{m}\right)}\right] = \frac{-\cancel{m}}{\cancel{2b}}\left[\log \frac{\left(g - \frac{bv_f^2}{m}\right)}{g}\right]$$

$$\Rightarrow g^2 = g^2 - \frac{b^2 v_0^2 v_f^2}{m^2} - \frac{gbv_f^2}{m} + g\frac{bv_0^2}{m}$$

$$\Rightarrow v_f^2 \frac{b}{m}\left(\frac{v_0^2}{m} + g\right) = g\frac{bv_0^2}{m}$$

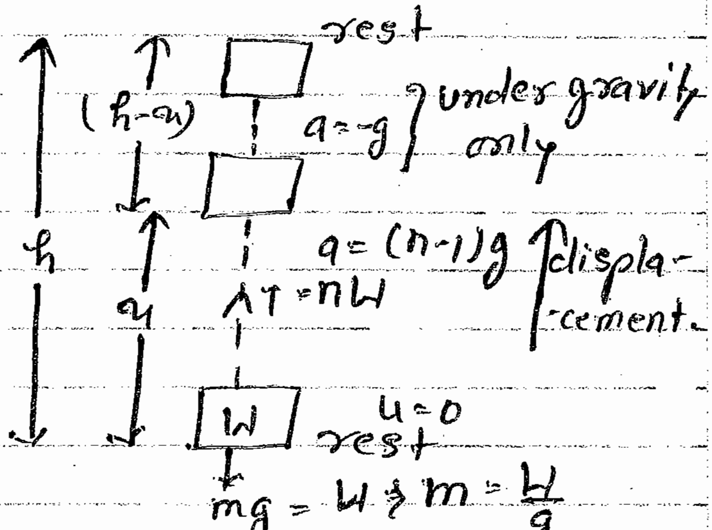
$$\Rightarrow v_f^2 = \frac{\cancel{g}\, v_0^2}{\cancel{g}\left(1 + \frac{v_0^2}{mg}\right)}$$

$$\Rightarrow \boxed{v_f = \frac{v_0}{\sqrt{\left(1 + \frac{v_0^2}{mg}\right)}}}$$

Ans

Que A load of weight W is to be raised by a rope from rest to rest through a height h. The greatest tension which the rope can safely bear is nW. The least time in which ascent can be made is -?

Sol^n Equation of motion -



$$nW - W = ma$$

$$W(n-1) = \frac{W}{g}a$$

$$\boxed{a = (n-1)g = \text{Constt.}}$$

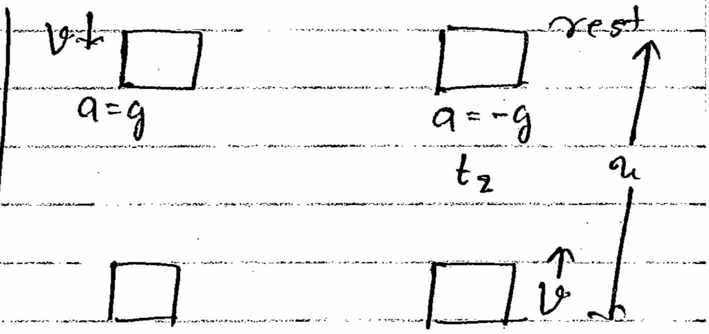
Let block is accelerated (pulled for a distance x)

Calculation of t_1 (time for which it is pulled)

$$s = ut + \frac{1}{2}at^2$$

$$x = 0 + \frac{1}{2}(n-1)g t_1^2$$

$$\boxed{t_1 = \sqrt{\frac{2x}{(n-1)g}}}$$



Time for up = Time for down

if only g acting

Calculation of time t_2 :-

$$s = ut + \frac{1}{2}at^2$$

$$(h-x) = 0 + \frac{1}{2}g t_2^2$$

$$\boxed{t_2 = \sqrt{\frac{2(h-x)}{g}}}$$

Total time :-

$$t = t_1 + t_2 = \sqrt{\frac{2}{g}}\left[\sqrt{\frac{x}{(n-1)}} + \sqrt{h-x}\right] \quad \text{—(1)}$$

for least time –

$$\frac{dt}{dx} = 0$$

So from (1)

$$\frac{dt}{dx} = \sqrt{\frac{2}{g}}\left[\frac{1}{\sqrt{n-1}} \times \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{(h-x)}}\right] = 0$$

Squaring.

$$\frac{1}{(n-1)x} = \frac{1}{h-x}$$

$$h - x = nx - x$$

$$\Rightarrow \boxed{x = \frac{h}{n}}$$

$\therefore$ for least time we put $x = \frac{h}{n}$

$$t = \sqrt{\frac{2}{g}}\left(\sqrt{\frac{h/n}{n-1}} + \sqrt{h - \frac{h}{n}}\right]$$

$$= \sqrt{\frac{2}{g}}\left[\sqrt{\frac{h}{n(n-1)}} + \sqrt{\frac{h(n-1)}{n}}\right]$$

$$= \sqrt{\frac{2h}{gn}}\left[\frac{1}{\sqrt{n-1}} + \sqrt{n-1}\right]$$

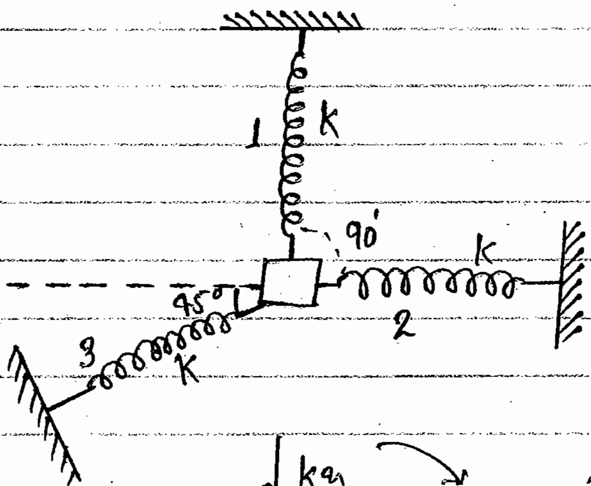
$$t = \sqrt{\frac{2h}{gn}} \times \frac{n}{\sqrt{n-1}}$$

$$\Rightarrow \quad t = \sqrt{\frac{2hn^2}{gn(n-1)}}$$

$$\Rightarrow \quad \boxed{t = \sqrt{\frac{2nh}{g(n-1)}}}$$

A-1
Q.3 In the figure shown the block is given a small displacement 'x' along the spring 3. Equation of the motion of the block is (Assume that system is lying on table).

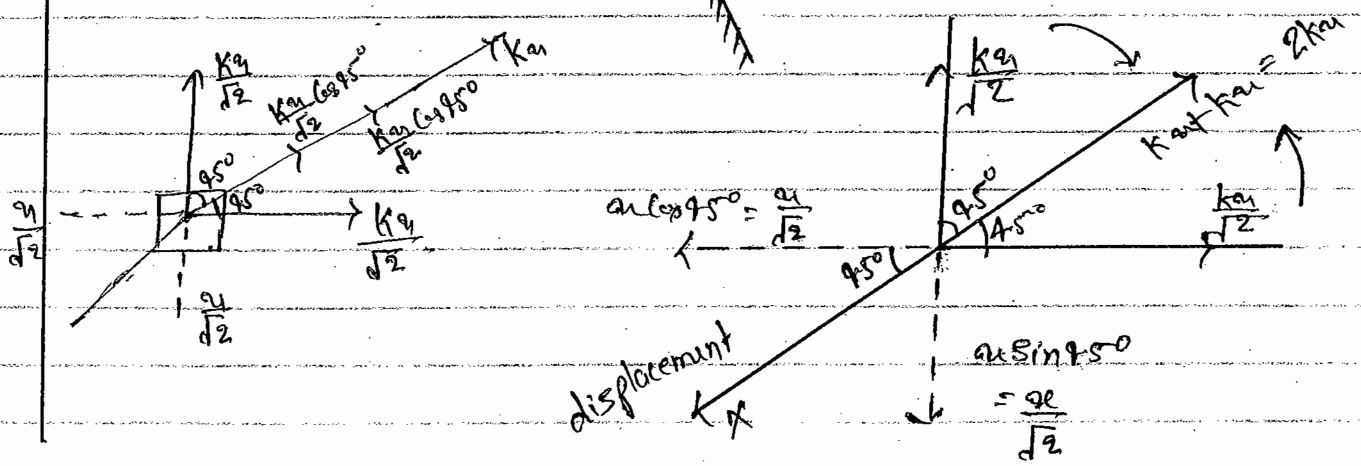
Sol^n Approximate method:-



$f_x = m a_x$

$-2kx = m\ddot{x}$

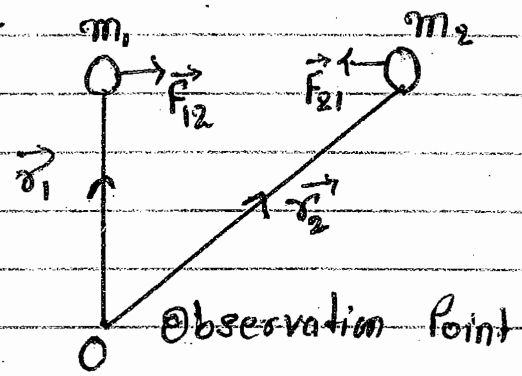
$$\boxed{\ddot{x} + \frac{2k}{m}x = 0}$$



Reduced Mass

When two objects are moving under their mutual interaction force. Then problem can be simplified by assuming that, one object is at rest. when we do so mass of other object had to be replaced by reduced mass.

Equation of motion of m_1 :-



$$\vec{F}_{12} = m_1 \frac{d^2\vec{r}_1}{dt^2}$$

$$\frac{\vec{F}}{m_1} \Leftarrow \frac{\vec{F}_{12}}{m_1} = \frac{d^2\vec{r}_1}{dt^2} \quad \text{—— (i)}$$

Equation of motion of m_2 :-

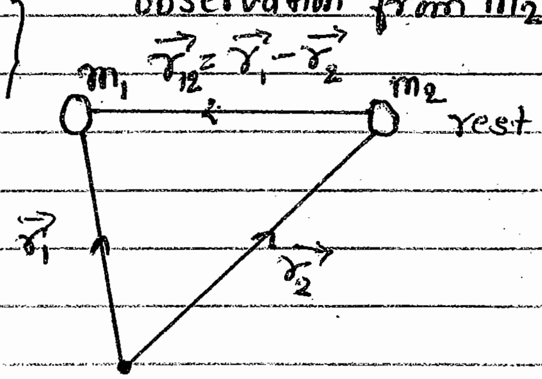
$$\frac{\vec{F}_{21}}{m_2} = \frac{d^2\vec{r}_2}{dt^2}$$

$$\Rightarrow \frac{-\vec{F}}{m_2} = \frac{d^2\vec{r}_2}{dt^2} \quad \text{—— (2)}$$

$\because$ from Newtons Law -

$$\left\{ \vec{F}_{12} = -\vec{F}_{21} = \vec{F} \text{ (let)} \right\}$$

observation from m_2



Subtracting (2) in (i)

$$\frac{\vec{F}}{m_1} + \frac{\vec{F}}{m_2} = \frac{d^2}{dt^2}(\vec{r}_1 - \vec{r}_2)$$

$$\vec{F}\,\frac{m_1 + m_2}{m_1 . m_2} = \frac{d^2\vec{r}_{12}}{dt^2}$$

$$\Rightarrow \vec{F} = \frac{m_1 m_2}{m_1 + m_2} \cdot \frac{d^2 \vec{r}_{12}}{dt^2}$$

$$\Rightarrow \vec{F} = \mu \, \vec{a}_{12}$$

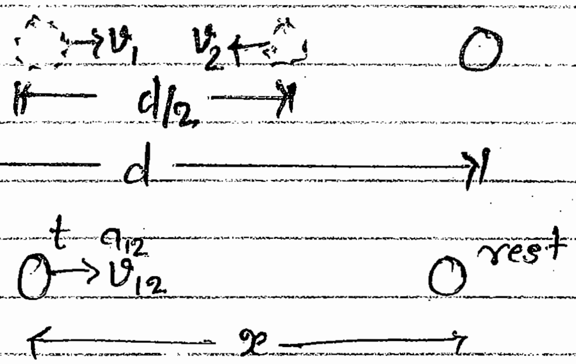
here $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

$$\Rightarrow \boxed{a_{12} = \frac{\vec{F}}{\mu}}$$

In this case velocity will be relative.

A-2
Q.7 Two particle of masses m_1 & m_2 are 'd' distance apart. Due to gravitational attraction they move towads each other. What is speed of m_1 when their separation reduces to $d/2$.

Sol^n:- Let m_2 is at rest.



$$a_{12} = \frac{F}{\mu} = \frac{G m_1 m_2 / x^2}{\frac{m_1 m_2}{m_1 + m_2}}$$

$$a_{12} = \frac{G(m_1 + m_2)}{x^2}$$

$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$

$\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$

$$-v_{12} \frac{dv_{12}}{dx} = \frac{G(m_1 + m_2)}{x^2}$$

{ ∵ here −ve sign is taken because x is decreasing }

$$-\int_0^{v_{12}} v_{12} \, dv_{12} = \int_d^{d/2} G(m_1 + m_2) \frac{dx}{x^2}$$

$$-\frac{v_{12}^2}{2} = -G(m_1 + m_2) \frac{1}{d}$$

$$v_{12} = \sqrt{\frac{2G(m_1 + m_2)}{d}}$$

$$v_1 + v_2 = \sqrt{\frac{2G(m_1 + m_2)}{d}} \quad \text{——— (1)}$$

Here no external force acts on system (m_1+m_2)

$\therefore \vec{P} = \text{Constant}$

$$\Rightarrow P_i = P_f \Rightarrow 0 = m_1 v_1 - m_2 v_2$$

$$\Rightarrow m_1 v_1 = m_2 v_2 \Rightarrow v_2 = \frac{m_1 v_1}{m_2} \quad \text{--- (ii)}$$

from (1) and (2)

$$v_1\left(1 + \frac{m_1}{m_2}\right) = \sqrt{\frac{2G(m_1+m_2)}{d}}$$

$$v_1 = \frac{m_2}{(m_1+m_2)} \sqrt{\frac{2G(m_1+m_2)}{d}}$$

$$\Rightarrow \boxed{v_1 = m_2 \sqrt{\frac{2G}{(m_1+m_2)d}}}$$

Conservative Forces

If work done by a force is independent of path followed, the force is called Conservative

⇒ Work done along closed path = zero.

$$\boxed{\oint \vec{F}\cdot d\vec{l} = 0} \qquad \{F \text{ is conservative}\}$$

Using stoke's theorem to convert into surface integral.

$$\int (\vec{\nabla}\cdot\vec{F})\cdot d\vec{s} = 0$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{F} = 0} \quad \text{— (i)}$$

from mathematics

$$\pm\,\vec{\nabla}\times\vec{\nabla}\,(\text{scalar}) = 0 \quad \text{— (ii)}$$

Calculating (i) & (ii)

$$F = \pm\vec{\nabla}\ \text{scalar}$$

Or

F from U → $\boxed{F = -\nabla U}$ — ve sign gives current direction of force.

$$U = P.E.$$

It is defined only for conservative forces.
If we fined U from F

Actual definition → $\boxed{U(r) = \int_{r}^{\text{ref. pt.}} \vec{F}\cdot d\vec{l}}$ Ref. pt. the pt. where P.E. is zero & also force is zero.

Gravitational Force [$F = mgh$ is used when $h <<< R_e$]

Short Method :-

$$U(x) = -\int F\,dx$$

$$U(r) = -\int F\,dr$$

If force is repulsive it is taken +ve
If force is attractive it is taken −ve.

Example of conservative forces :-

1. Gravitational force b/w two masses :-

$$\boxed{F = \frac{G\,m_1 m_2}{r^2}} \quad \text{(attractive)}$$

{ −ve sign is taken because force is attractive

$$U = -\int F.dr = -\int \frac{-G m_1 m_2}{r^2}\,dr = -\frac{G m_1 m_2}{r}$$

2. Potential Energy of Earth Mass System :-

$$U = -\frac{G M_e m}{R_e + h}$$

P.E. = 0 at surface with $h = 0$

$$\Rightarrow \frac{-G M_e m}{R_e\left(1 + \frac{h}{R_e}\right)} = -\frac{G M_e m}{R_e}\left(1 + \frac{h}{R_e}\right)^{-1}$$

P.E. = U_{ref} $= -\frac{G M_e m}{R_e}$

$$\Rightarrow U = -\frac{G M_e m}{R_e}\left(1 - \frac{h}{R_e}\right)$$

$$\Rightarrow U = -\frac{G M_e m}{R_e} + \frac{G M_e m}{R_e^2} h$$

{ $\because U_{ref} = -\frac{G M_e m}{R_e}$, $g = \frac{G M_e}{R_e^2}$

$$\Rightarrow U = U_{ref} + mgh$$

(true for small h) $h <<< R_e$

$$\Rightarrow U - U_{ref} = mgh$$

{ If $h = 2R_e$ or $\frac{R_e}{2}$ etc then this relation will not be used }

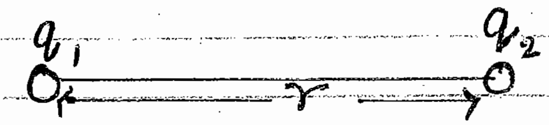
To convert electrostatic formulae into Gravitational Formulas
Replace $q \longrightarrow m$, & $\frac{1}{4\pi\epsilon_0} \longrightarrow G$

---

if $U_{ref} = 0$

Then, $\boxed{U = mgh}$

3. Electrostatic force :- {It is a conservative force.}

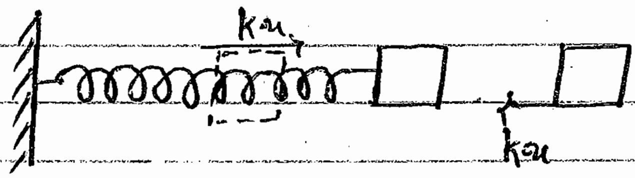


$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

$q_1 > 0$, $q_2 > 0$

$$U = -\int F.\,dr \Rightarrow \boxed{U = \frac{q_1 q_2}{4\pi\epsilon_0 r}}$$

4. Spring force {Conservative} :-



$$U(x) = -\int F\,dx$$

$$= -\int (-kx)\,dx$$

$$\boxed{U(x) = \frac{1}{2}kx^2}$$

* Conservation of Mechanical Energy :-

If only conservative forces are acting on a system then total mechanical energy of the system remains conserved.

$$\boxed{\text{Total Energy} = \text{Constant}}$$

$$K.E. + P.E. = \text{Constant}$$

$$(K.E. + P.E.)_i = (K.E. + P.E.)_f$$

Mechanical energy is not conserved if non conservative forces act on the system.

Non-Conservative forces → friction and any other dissipative forces.

{If there is other than above three forces then we use Work Energy Theorem}.

* **Work Energy Theorem** :- Total work done by all forces is equal to change in K.E.

$$\boxed{\text{Total Work Done} = K.E_f - K.E_i}$$

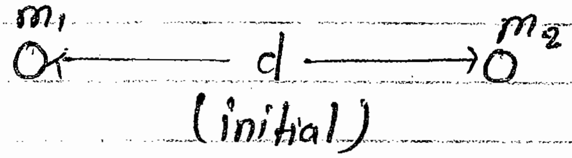
II^{nd} Method.

A-2

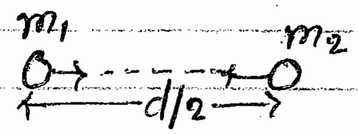
Ques 7 Two particles of masses m_1 & m_2 are placed 'd' distance apart. Due to gravitational attraction they move towards each other. What is speed of m_1 when their separation reduces to $d/2$.

Sol^n:

Use conservation of mechanical energy:-



$$(K.E. + P.E.)_i = (K.E. + P.E.)_f$$



$$0 - \frac{G m_1 m_2}{d} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{G m_1 m_2}{d/2}$$

$$\frac{G m_1 m_2}{d} = \frac{1}{2}\left[m_1 v_1^2 + m_2 v_2^2\right] \quad \text{—— (I)}$$

from Conservation of momentum:-

$$P_{initial} = P_{final}$$

$$0 = m_1 v_1 - m_2 v_2 \Rightarrow v_2 = \frac{m_1 v_1}{m_2} \quad \text{—— (II)}$$

Put v_2 in (I) -

$$\frac{G m_1 m_2}{d} = \frac{1}{2}\left[m_1 v_1^2 + \frac{m_1^2 v_1^2}{m_2}\right]$$

$$= \frac{1}{2} m_1 v_1^2 \left[1 + \frac{m_1}{m_2}\right]$$

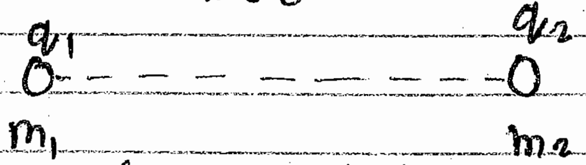
$$m_1 \left(\frac{m_1 + m_2}{m_2}\right) v_1^2 = \frac{2 G m_1 m_2}{d}$$

$$v_1 = \sqrt{\frac{2G \cancel{m_1} m_2^2}{\cancel{m_1}(m_1+m_2)d}} \Rightarrow \boxed{v = \sqrt{\frac{2G m_2^2}{(m_1+m_2)d}}}$$

Note:-

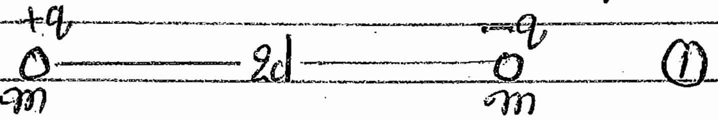
When two or many Particle having mass as well as Charge :-

$\because$ $F_{gravitational} << F_{coulombic}$

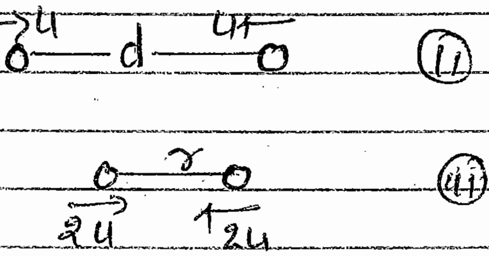


$\therefore$ Neglect effect of gravitation in comparision to electrostatic force.

Ques 5 Two particles of equal masses & having opposite charges are placed 2d distance apart. Due to electrostatic force they move towards each other. When separation between them reduces to 'd' their Speeds becomes 'u'. At what separation speed of each particle is '2u'.



$(K.E. + P.E.)_{(i)} = (K.E. + P.E.)_{(ii)}$



$$0 + \frac{q^2}{4\pi\epsilon_0 (2d)} = \cancel{2} \times \frac{1}{\cancel{2}} mu^2 - \frac{q^2}{4\pi\epsilon_0 d}$$

$$\frac{q^2}{8\pi\epsilon_0 d} = mu^2 \quad \text{--- (1)}$$

Now

$(K.E. + P.E.)_{(i)} = (K.E. + P.E.)_{(iii)}$

$$0 + \frac{q^2}{8\pi\epsilon_0 d} = \cancel{2} \times \frac{1}{\cancel{2}} m(2u)^2 - \frac{q^2}{4\pi\epsilon_0 r}$$

$$\frac{-q^2}{8\pi\epsilon_0 d} = 4mu^2 - \frac{q^2}{4\pi\epsilon_0 r}$$

Put value of mu^2 from (1)

$$\Rightarrow \frac{-\cancel{q^2}}{8\pi\epsilon_0 d} = \frac{4\cancel{q^2}}{8\pi\epsilon_0 d} - \frac{\cancel{q^2}}{4\pi\epsilon_0 r}$$

$$\Rightarrow \frac{-1}{\cancel{8\pi\epsilon_0} d} = \frac{4}{\cancel{8\pi\epsilon_0} d} - \frac{2\times 1}{\cancel{2\times 4\pi\epsilon_0} r}$$

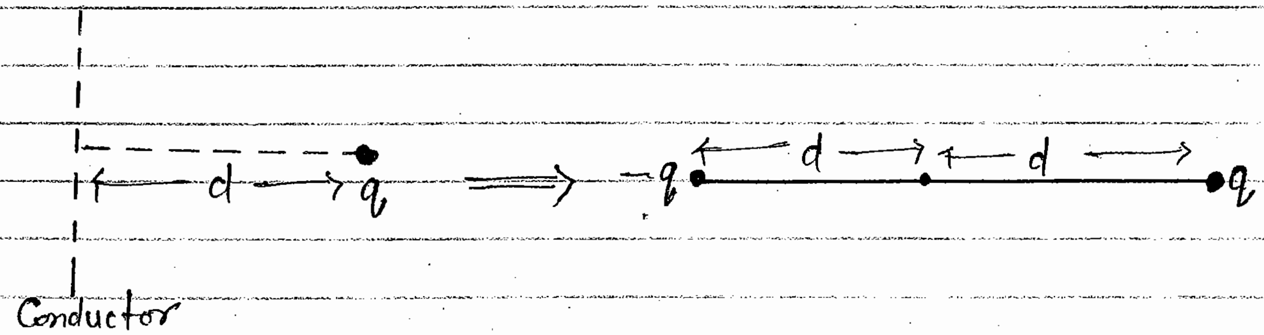
$$\Rightarrow \frac{-1}{d} = \frac{4}{d} - \frac{2}{r}$$

$$\Rightarrow \frac{\cancel{+}5}{d} = \frac{\cancel{+}2}{r}$$

$$\boxed{r = \frac{2d}{5}}$$

Note:-

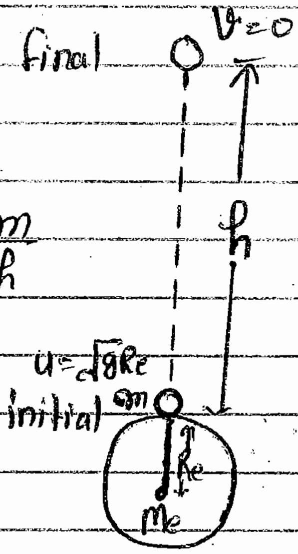
If a conductor is placed at a distance d from a charge q, then we can remove conductor by a charge $-q$, putting at a distance $2d$ from q. i.e.



→ In a question if speed is given in formula form then we consider speed large.

Q. A particle is thrown upward vertically with initial speed $\sqrt{gR_e}$ where g is acceleration due to gravity on earth's surface. What is the maximum height attain by the particle?

Sol^n



$(K.E. + P.E.)_i = (K.E. + P.E.)_f$

$$\frac{1}{2}m\left(\sqrt{gR_e}\right)^2 - \frac{GM_em}{R_e} = 0 - \frac{GM_em}{R_e+h}$$

$$\Rightarrow \frac{1}{2}\frac{\cancel{GM_e}\cancel{R_e}}{R_e^{\cancel{2}}} - \frac{\cancel{GM_e}}{R_e} = -\frac{\cancel{GM_e}}{R_e+h}$$

$$\Rightarrow \frac{1}{2R_e} - \frac{1}{R_e} = \frac{-1}{R_e+h}$$

$$\Rightarrow \frac{\cancel{-}1}{2R_e} = \frac{\cancel{-}1}{R_e+h}$$

$$\Rightarrow R_e + h = 2R_e$$

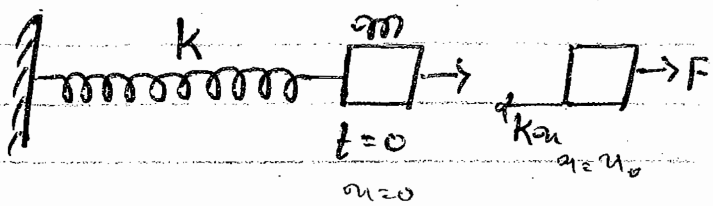
$$\Rightarrow \boxed{h = R_e}$$

Ans

Basic Assignment

Ques 5 In the fig. shown the block is pulled with a constt. force F. What is speed of the block at the instant when acc^n is zero?

Sol^n Equation of motion :-



$$F - kx = ma \quad \text{--- (1)}$$

Let $a = 0$ when $x = x_0$

$$F - kx_0 = 0$$

$$\Rightarrow \boxed{x_0 = \frac{F}{k}}$$

Now from (1)

$$F - kx = mv\frac{dv}{dx}$$

$$\int_0^{F/k} (F - kx)\,dx = m\int_0^v v\,dv$$

$$\left[Fx - \frac{kx^2}{2}\right]_0^{F/k} = \frac{mv^2}{2}$$

$$\frac{F^2}{k} - \frac{F^2}{2k} = \frac{mv^2}{2}$$

$$\Rightarrow \frac{F^2}{\cancel{2}k} = \frac{mv^2}{\cancel{2}}$$

$$\boxed{v = \frac{F}{\sqrt{mk}}}$$

Second Method :-

Since there is another force so we can use work energy theorem.

Work Energy Theorem,

$$T.W.D. = K.E_f - K.E_i$$

$$W_S + W_F = \frac{1}{2}mv^2 - 0$$

$$\int_0^{F/k} \vec{F}_{spring} \cdot d\vec{l} + \int_0^{F/k} \vec{F} \cdot d\vec{l} = \frac{1}{2}mv^2$$

$$\int_0^{F/k} kx\,dx \cos 180^\circ + \int_0^{F/k} F\,dx \cos 0^\circ = \frac{1}{2}mv^2$$

$$\left[-\frac{1}{2}kx^2\right]_0^{F/k} + F \cdot \frac{F}{K} = \frac{1}{2}mv^2$$

$$\Rightarrow -\frac{1}{2}\cancel{K}\frac{F^2}{K^2} + \frac{F^2}{K} = \frac{1}{2}mv^2$$

$$\Rightarrow \frac{1}{2}\frac{F^2}{K} = \frac{1}{2}mv^2$$

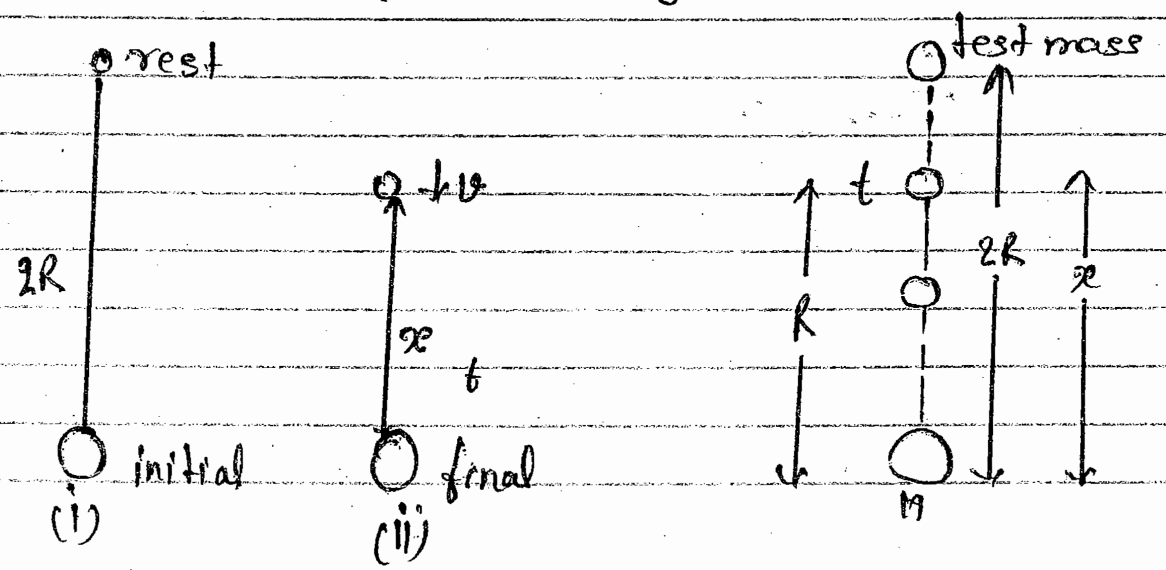
$$\Rightarrow v^2 = \frac{F^2}{mK}$$

$$\Rightarrow \boxed{v = \frac{F}{\sqrt{mK}}}$$

Ans

A-1 (Method-II)

Q.10 The free fall time of a test mass on an object of mass M from a hight 2R to R is -



Apply Conservation of energy.

$$(K.E. + P.E.)_I = (K.E. + P.E.)_{II}$$

$$0 + \frac{GM\cancel{m}}{2R} = \frac{1}{2}\cancel{m}v^2 - \frac{GM\cancel{m}}{x}$$

$$GM\left(-\frac{1}{2R} + \frac{1}{x}\right) = \frac{1}{2}v^2$$

$$v = \sqrt{GM}\sqrt{\frac{2}{x} - \frac{1}{R}}$$

$$\Rightarrow \frac{-dx}{dt} = \sqrt{GM}\sqrt{\frac{2R-x}{xR}}$$ $\left\{ v = \frac{-dx}{dt} \right.$ is used because x is decreasing.

$$\Rightarrow \frac{-dx}{dt} = \sqrt{\frac{GM}{R}}\sqrt{\frac{2R-x}{x}}$$

$$\Rightarrow -\int_{2R}^{R} \frac{\sqrt{x}\,dx}{\sqrt{2R-x}} = \sqrt{\frac{GM}{R}}\int_0^t dt \quad \text{---} \; (1)$$

Put $x = 2R\sin^2\theta \Rightarrow \sin\theta = \sqrt{\frac{x}{2R}} \Rightarrow \theta = \sin^{-1}\sqrt{\frac{x}{2R}}$

$dx = 4R\sin\theta\cos\theta\,d\theta$

Integrating -

$$\int_{2R}^{R} \frac{\sqrt{x}}{\sqrt{2R-x}}\,dx = \int_{2R}^{R} \frac{\sqrt{\cancel{2R}}\sin\theta \, 4R\sin\theta\cancel{\cos\theta}\,d\theta}{\sqrt{\cancel{2R}}\cancel{\cos\theta}}$$

$$\Rightarrow 2R\int 2\sin^2\theta\,d\theta = 2R\int(1-\cos 2\theta)\,d\theta$$

$$\Rightarrow 2R\left[\theta - \frac{\sin 2\theta}{2}\right]_{2R}^{R} = 2R\left[\theta - \sin\theta\cos\theta\right]$$

$$\Rightarrow 2R\left[\sin^{-1}\sqrt{\frac{x}{2R}} - \sqrt{\frac{x}{2R}}\sqrt{1-\frac{x}{2R}}\right]_{2R}^{R} = 2R\left[\frac{\pi}{4} - \frac{\pi}{2} - \frac{1}{2} + 0\right]$$

$$\Rightarrow 2R\left[-\frac{\pi}{4}-\frac{1}{2}\right] = -R\left[\frac{\pi}{2}+1\right]$$ {Putting this value of integration in (1)

$$\Rightarrow -\left\{-R\left[\frac{\pi}{2}+1\right]\right\} = \sqrt{\frac{GM}{R}}\,t$$

$$\Rightarrow t = R\left[\frac{\pi}{2}+1\right]\sqrt{\frac{R}{GM}}$$

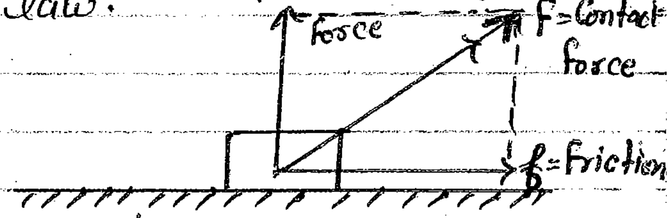
$$\Rightarrow \boxed{t = \left(\frac{\pi}{2}+1\right)\sqrt{\frac{R^3}{GM}}}$$

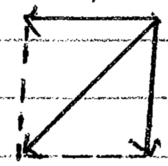
Ans

{Friction Force}

When two bodies are kept in contact, electro-magnetic forces act between the charged particles at the surfaces of the bodies. As a result each body exerts a contact force on the other.

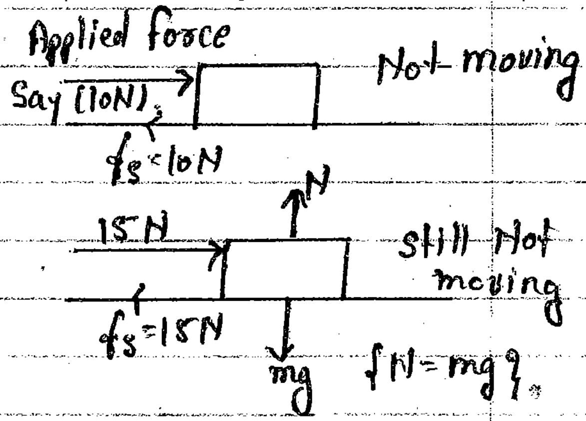
The magnitude of the contact forces on the two bodies are equal but their direc^n are opposite and hence the contact forces obey Newton's III^rd law.





Static Friction (f_s) :-

When two bodies do not slip on each other the force of friction is called static friction.



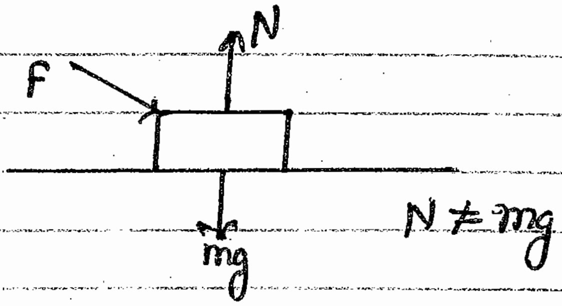
$\therefore$ $\boxed{f_s \leq \mu_s N}$

N → Normal Reaction

μ_s → Coefficient of static friction.

The limiting value of static friction is -

$$\boxed{(f_s)_{max} = \mu_s N}$$

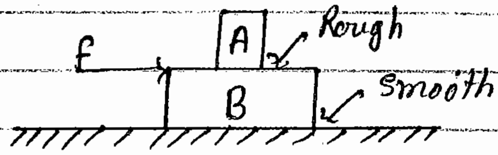


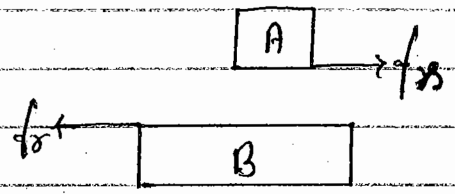
N may or may not be equal to mg.

$N \neq mg$

⇒ friction may not be opposite to direction of motion always.

Ex- Here A will move with B since B is applying friction on A.



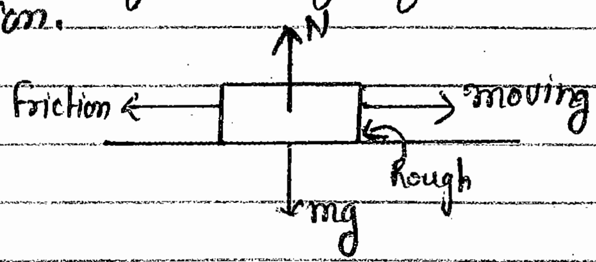


by Newton's Law.

Kinetic Friction :- (f_k) :-

When two solid bodies slip over each other, the force of friction is called kinetic friction.

$$\boxed{f_k = \mu_k N}$$



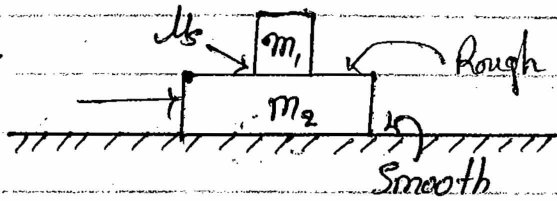
$\mu_k < \mu_s$ (slightly)

- Maximum force that can be applied without moving an object is $\mu_s N$.
- Minimum force required to move an object is almost equal to $\mu_k N$.

* Two Object Cases :-

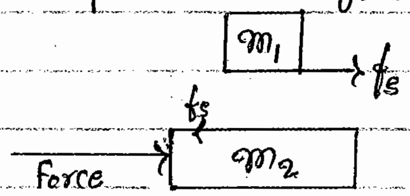
Q. What maximum force can be applied on lower block so that the two blocks move together (no relative motion between them).

Sol^n



The acceleration of upper block has a limit because it is moving due to static friction.

{Upper block will move due to static friction



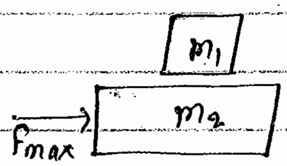
Maximum acceleration of upper block $= \frac{(f_s)_{max}}{m_1}$

$$= \frac{\mu_s N}{m_1}$$

$$= \frac{\mu_s \cancel{m_1} g}{\cancel{m_1}} = \mu_s g.$$

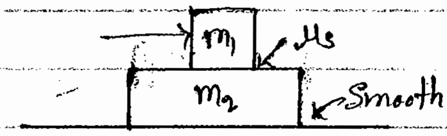
The two blocks will move together without any relative motion if their acceleration is less or equal to $\mu_s g$.

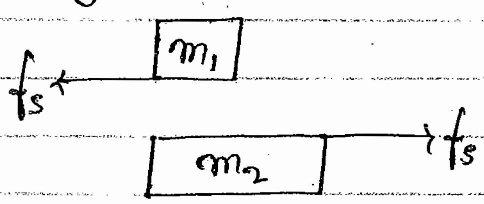
$$\boxed{F_{max} = (m_1 + m_2)\mu_s g}$$



$a_{max} = \mu_s g$

Q. What is maximum acc^n so that the two blocks move together.



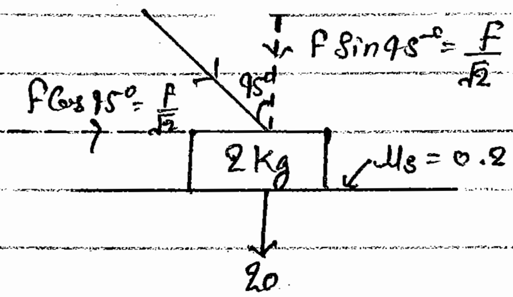


$$\text{Maximum acc}^n \text{ of } m_2 = \frac{(f_s)_{max}}{m_2} = \frac{\mu_s N}{m_2}$$

$$\boxed{\text{Maximum acc}^n \text{ of } m_2 = \frac{\mu_s m_1 g}{m_2}}$$

Basic.
Q.15 A 2 kg block is lying on a rough surface. If coefficient of static friction between the block & ground is 0.2. What max^m force can be applied on the block without moving it.

Sol^n



The block will not move if –

$$\frac{F}{\sqrt{2}} = (f_s)_{max} = \mu_s N$$

$$\frac{F}{\sqrt{2}} = 0.2 N \quad \text{—— (1)}$$

Vertical Equilibrium :-

$$20 + \frac{F}{\sqrt{2}} = N$$

Put N from (1)

$$20 + \frac{F}{\sqrt{2}} = \frac{F}{0.2\sqrt{2}}$$

$$\frac{F}{\sqrt{2}}\left[1 - \frac{1}{0.2}\right] = -20 \Rightarrow -\frac{F}{\sqrt{2}}\left[\frac{0.8}{0.2}\right] = -20$$

$$\Rightarrow F = \frac{20\sqrt{2}}{4} \Rightarrow F = 5\sqrt{2}$$

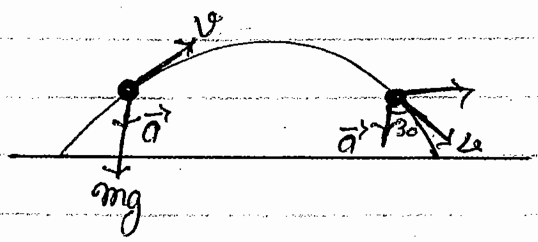
$$\Rightarrow \boxed{F \approx 7N} \quad \underline{\text{Ans}}$$

To match the option

Basic

Q.10 A projectile is thrown at some angle. What is tangential acc^n of projectile at the instant its velocity vector makes 30° with its acc^n vector.

Sol^n ∵ Direction of acc^n will be towards force direction



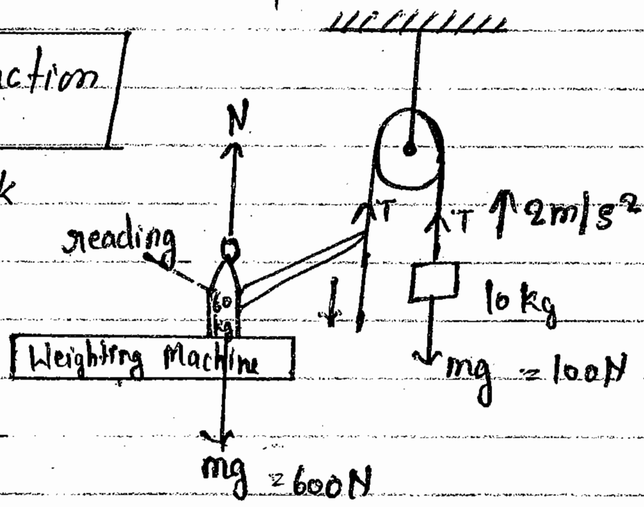
$$\vec{a} = \frac{\vec{F}}{m} = \frac{mg}{m} = g$$

$$a_t = a \cos 30°$$

$$\boxed{a_t = g\frac{\sqrt{3}}{2}}$$ Ans

Q.11 A person standing on a weighting machine as shown in the fig. pulls the string attached to a block in vertically downward direction. What is reading shown by the machine.

Sol^n



$\boxed{\text{Reading} = \text{Normal Reaction}}$

Eq^n of motion of block

$$T - mg = ma$$

$$T - 100 = 10 \times 2$$

$$T = 120$$

Consider equilibrium of person

$$N + T = mg \Rightarrow N = Mg - T$$

$$= 600 - 120$$

$$= 480 \text{ Newton}$$

$$\boxed{N = 48 \text{ kg}}$$ Ans.

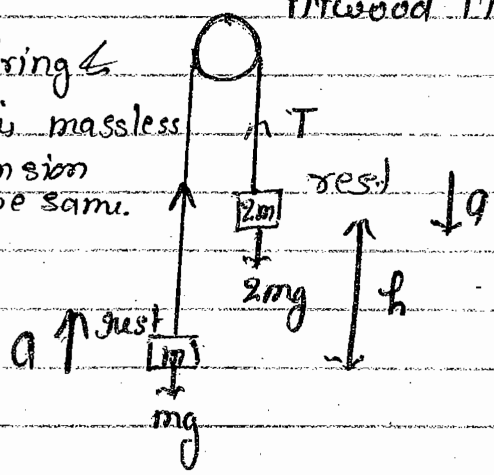
Q.7 In the fig. shown the two blocks are released from the position shown. After what time the two will cross each other. [Assume pulley & string to be light & smooth].

Sol^n

Equation of motion for m :- Atwood Machine.

$$T - mg = ma \quad \text{--- (1)}$$

{ If string & pulley is massless then tension will be same.



for 2m :-

$$2mg - T = 2ma \quad \text{--- (2)}$$

adding (1) & (II)

$$\cancel{m}g = 3\cancel{m}a$$

$$\boxed{a = \frac{g}{3}}$$

$$S = ut + \frac{1}{2}at^2$$

$$\frac{h}{2} = 0 + \frac{1}{2}\frac{g}{3}t^2$$ { The distance for crossing will be $h/2$

$$t^2 = \frac{3h}{g}$$

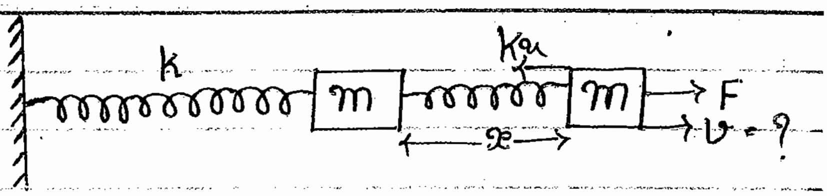
$$\Rightarrow \boxed{t = \sqrt{\frac{3h}{g}}} \text{ Ans}$$

A-3

Q.12 A spring mass system (Spring constant k & mass m) lies on a smooth horizontal surface with one end of spring being rigidly fixed. At t = 0 the mass is pulled with a constant horizontal force F. Speed of the mass after time t is ?

Sol^n

Equation of motion -



$$\vec{F} - kx = m\vec{a}$$

$$\boxed{\vec{a} = \frac{F}{m} - \frac{k}{m}x}$$ $\because$ $\vec{a}$ is variable

$\because$ We want to calculate - $v = f(t)$

So $F - kx = m\frac{dv}{dt}$

Steps :-

$$\boxed{\vec{a} = v\frac{dv}{dx} \xrightarrow{\text{integrating}} v = f(x) \xrightarrow{\text{integrate}} x = f(t) \xrightarrow{\text{differentiating}} v = f(t)}$$

So

$$F - kx = mv\frac{dv}{dx}$$

$$\frac{(F - kx)}{m}dx = v\,dv$$

$$\int \frac{F}{m}dx - \int \frac{k}{m}x\,dx = \int v\,dv$$

$$\frac{Fx}{m} - \frac{kx^2}{2m} = \frac{v^2}{2}$$

$$v^2 = \frac{2Fx}{m} - \frac{kx^2}{m} = \frac{2Fx - kx^2}{m} = \frac{2}{m}\left[Fx - \frac{kx^2}{2}\right]$$

$$v = \sqrt{\frac{2x\left[F - kx/2\right]}{m}}$$

$$\frac{dx}{dt} = \sqrt{\frac{2x\left[F - kx/2\right]}{m}}$$

$$\frac{dx}{\sqrt{2Fx - \frac{2kx^2}{2}}} = \frac{1}{\sqrt{m}}\int dt$$

Now integrating

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}$$

$$\Rightarrow \int \frac{dx}{\sqrt{[2Fx - kx^2]}} = \frac{1}{\sqrt{m}}\int dt$$

let $2Fx - kx^2 = -[kx^2 - 2Fx]$

$$= -\left[(\sqrt{k}x)^2 - 2\sqrt{k}x \cdot \frac{F}{\sqrt{k}} + \left(\frac{F}{\sqrt{k}}\right)^2 - \left(\frac{F}{\sqrt{k}}\right)^2\right]$$

$$= -\left[\left(\sqrt{k}x - \frac{F}{\sqrt{k}}\right)^2 - \left(\frac{F}{\sqrt{k}}\right)^2\right]$$

$$= \left(\frac{F}{\sqrt{k}}\right)^2 - \left(\sqrt{k}x - \frac{F}{\sqrt{k}}\right)^2$$

$$(a)^2 \qquad\qquad (y)^2$$

$\sqrt{k}x - \frac{F}{\sqrt{k}} = y$

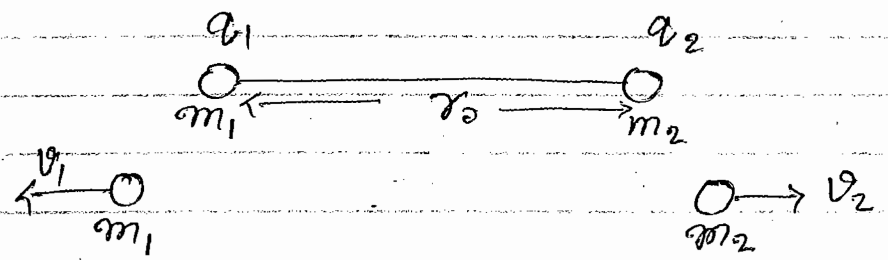
$\sqrt{k}\,dx = dy$

$dx = \frac{dy}{\sqrt{k}}$

Q.5 Two particles of masses m_1, m_2 and charges q_1, q_2 are placed r_0 distance apart on a smooth horizontal surface. Due to electrostatic repulsion they move away from each other. Ratio of their kinetic energy at a later time is -

Sol^n

$\vec{F}_{ext} = 0$



$$0 = m_2 V_2 - m_1 V_1$$

$$m_2 V_2 = m_1 V_1$$

$$\boxed{\frac{V_1}{V_2} = \frac{m_2}{m_1}} \text{ ——— (1)}$$

$$\frac{K_1}{K_2} = \frac{\cancel{\frac{1}{2}} m_1 V_1^2}{\cancel{\frac{1}{2}} m_2 V_2^2}$$

$$\frac{K_1}{K_2} = \frac{m_1}{m_2}\left(\frac{V_1}{V_2}\right)^2$$

$$\frac{K_1}{K_2} = \frac{m_1}{m_2}\left(\frac{m_2}{m_1}\right)^2$$

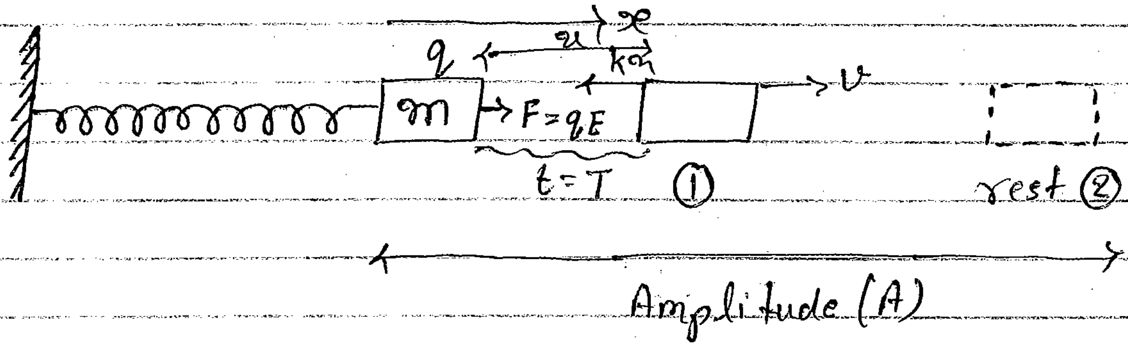
$$\boxed{\frac{K_1}{K_2} = \frac{m_2}{m_1}}$$

Ans

Net - 2011

2.10 A point particle of mass m carrying an electric charge q is attached to a spring of stiffness constant K. A constant electric field E along the direction of spring is switched on for a time interval T (where $T \ll \sqrt{\frac{m}{K}}$). Neglecting radiation loss, the amplitude of oscillation after the field is switched off is -

Sol^n



$$v = \frac{qE}{\sqrt{mk}} \sin\sqrt{\frac{K}{m}}\,T$$

$$\frac{dx}{dt} = \frac{qE}{\sqrt{mk}} \sin\sqrt{\frac{K}{m}}\,t$$

$$[x]_0^x = \frac{qE}{\sqrt{mk}}\left[-\cos\sqrt{\frac{K}{m}}\,t\right]_0^T \times \sqrt{\frac{m}{K}}$$

Elongation $\boxed{x = \frac{qE}{K}\left[1 - \cos\sqrt{\frac{K}{m}}\,T\right]}$

Conservation of energy:-

$$(K.E. + P.E.)_I = (K.E. + P.E.)_{II}$$

$$\left(\cancel{\tfrac{1}{2}} m v^2 + \cancel{\tfrac{1}{2}} k x^2\right) = \left(0 + \cancel{\tfrac{1}{2}} k A^2\right)$$

Date
31/07/2019

$$\Rightarrow m.\frac{q^2E^2}{mk}\sin^2 + \frac{k\,q^2E^2}{k^2}(1-\cos)^2 = kA^2$$

$$\Rightarrow \frac{q^2E^2}{k}\left[\sin^2 + 1 + \cos^2 - 2\cos\right] = kA^2$$

$$\Rightarrow 2\frac{q^2E^2}{k^2}\left[1-\cos\sqrt{\frac{k}{m}}\,T\right] = A^2$$

$$\Rightarrow \frac{4q^2E^2}{k^2}\left[\sin^2\sqrt{\frac{k}{m}}\cdot\frac{T}{2}\right] = A^2$$

$$\Rightarrow A = \frac{2qE}{k}\sin\sqrt{\frac{k}{m}}\,\frac{T}{2}$$

$$\Rightarrow A = \frac{2qE}{k}\sin\frac{T}{2\sqrt{\frac{m}{k}}} \longrightarrow \text{Very small}$$

$$\Rightarrow A = \frac{\cancel{2}qE}{k}\,\frac{T}{\cancel{2}\sqrt{\frac{m}{k}}}$$

$$= \frac{qE}{\cancel{k}}\sqrt{\frac{\cancel{k}}{m}}\,T$$

$$= \frac{qE}{\sqrt{km}}\,T$$ Ans

~~A-3 Ques In previous question if m = Me, then time of fall will be -~~

A-3

Q.7 A small object of mass m falls from a height equal to radius of earth Re. If Me be mass of earth time taken by the particle to reach the earth's surface is (take $m << Me$).

(a) $\left(\frac{\pi}{2}+1\right)\sqrt{\frac{2R_e^3}{GM_e}}$ (b) $\left(\frac{\pi}{2}+1\right)\sqrt{\frac{R_e^3}{GM_e}}$

(c) $\frac{\pi}{2}\sqrt{\frac{R_e^3}{GM}}$ (d) $\sqrt{\frac{2R_e^3}{GM}}$

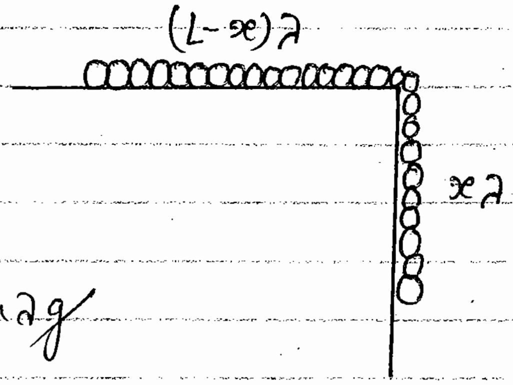
Sol^n

Basic

Ques A uniform rope of length 'L' is hanging off the edge of a rough table having coefficient of static friction μ. What should be minimum length of hanging hanging part so that the rope starts sliding down.

(a) $\frac{\mu L}{2}$ (b) $\frac{\mu L}{\mu+1}$ ✓ (c) $\left(\frac{1}{\mu}-1\right)L$ (d) $\frac{\mu L}{2-\mu}$

Sol^n



Equation of motion:-

$$x\lambda g = \mu (L-x)\lambda g$$

$$\Rightarrow x\lambda \cancel{g} = \mu L\lambda \cancel{g} - \mu x\lambda \cancel{g}$$

$$\Rightarrow x\cancel{\lambda} = \mu L\cancel{\lambda} - \mu x\cancel{\lambda}$$

$$\Rightarrow x + \mu x = \mu L$$

$$\Rightarrow x(1+\mu) = \mu L$$

So $\boxed{x = \frac{\mu L}{1+\mu}}$ Ans

31/July/2014

Stability Analysis

By stability analysis we mean finding equilibrium positions and investigating whether the given equilibrium is stable or unstable. It is easier to do stability analysis through potential rather than force. Therefore we will try to write potential of given system to discuss equilibrium whenever required.

Equilibrium criteria in one dimension :-

In such problem potential energy is given. If $V(x)$ be the potential under which a particle is moving then force acting on the particle is

$$\boxed{\vec{F_x} = -\frac{dV(x)}{dx}}$$

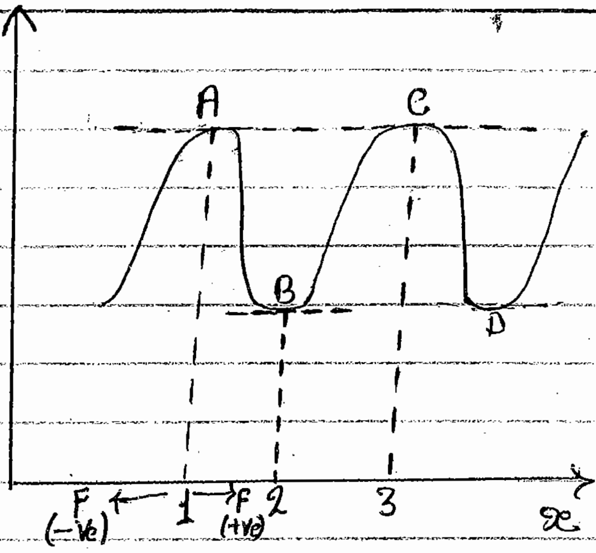
If system (Particle) is in equilibrium, then net force on particle is zero.

So, $\vec{F_n} = 0$

$$\Rightarrow \boxed{\frac{dV(x)}{dx} = 0}$$ (It gives equilibrium point (positions))

In $V(x)$ versus x graph $\frac{dV}{dx} = 0$ at the points where tangent to the curve is parallel to x-axis. Therefore in the figure shown below point A, B, C are equilibrium points.

Point 1, 2, and 3 are equilibrium points in space.



Point A is the point where slope = 0
" B " " " = 0
" B " " " = 0

Stable Equilibrium Point:-

A point is stable equilibrium point if, a particle at this point when displaced towards right experiences force towards left and vice-versa. That is the force tries to bring it back.
Therefore at stable equilibrium point -

$$\frac{dF_x}{dx} < 0$$

or

$$\boxed{\frac{dF_x}{dx} = -ve}$$

$\therefore \frac{d^2V(x)}{dx^2} > 0$ (Condition for minimum)

Thus, a stable equilibrium point is a minimum on V(x) versus x plot.
Therefore points B and D are stable point.

Unstable Equilibrium point:-

In this case a particle at equilibrium point is when displaced ~~towards~~ towards right it experiences force

also in rightward direction. That is the force tries to displace the particle away from equilibrium point. Therefore at unstable equilibrium point,

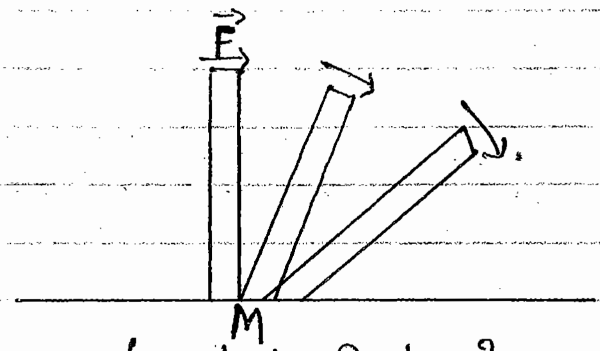
$$\frac{dF_x}{dx} > 0 \quad \text{or}$$

$$\boxed{\frac{dF_x}{dx} = +ve}$$

$$\therefore \quad \frac{d^2V(x)}{dx^2} < 0 \qquad \text{(Condition of maximum)}$$

Thus, an unstable point is a maximum on $V(x)$ versus x plot. Therefore points A and C in the above figure are unstable points.

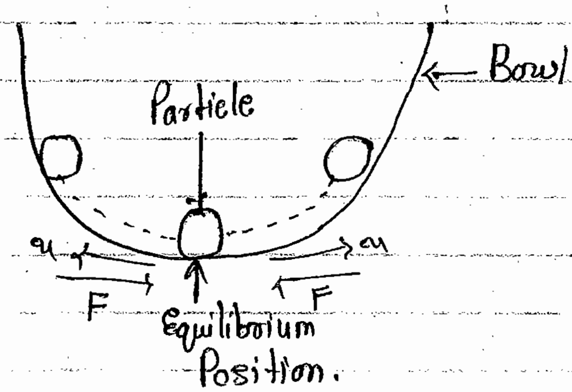
Example :-



{Unstable Position}

To understand equilibrium more closely we consider another example we take a hemispherical bowl and a marble like small sphere as shown in figure.

He we can see that the direction of force is opposite to the direction of displacement.



Now from $V(x)$ v/s x graph :-

⇒ At point 1 :-

Rightwards displacement :-

$$dx = +ve \quad , \quad dV = -ve$$

So $$\vec{F} = -\frac{dV}{dx} = +ve$$

So force is positive.

Leftwards displacement :-

$$dx = -ve \quad , \quad dV = -ve$$

$$\vec{F} = -\frac{dV}{dx} = +ve$$

So leftward displacement $= -ve$

Here

~~Rightwards Force ≠ Leftwards Force~~

Therefore we conclude that point 1 is unstable point.

∴ At unstable point $\frac{d^2V}{dx^2} < 0$ {for max^m V}

A condition of maxima.

⇒ At point 2 in $V(x)$ V/s x graph :-

Rightwards displacement :-

$$dV = +ve$$
$$dx = +ve$$

$$\vec{F} = -\frac{dV}{dx} = -ve$$

So for +ve displacement force is negative.

* Leftwards Displacement :-

for $dx = -ve$
$dv = +ve$

$$\vec{F} = -\frac{dV}{dx} = +ve$$

So for negative displacement we get force is +ve

So we conclude that point 2 is stable point.
i.e. At stable point we must have -

$$\boxed{\frac{d^2V}{dx^2} > 0} \qquad \{\text{for minimum } V\}$$

* Special Condition :-

If $\frac{d^2V}{dx^2} = 0$ then check higher even derivative. Then -

If,

$$\frac{d^nV}{dx^n} > 0 \longrightarrow (\text{Stable})$$

when

$$\frac{d^nV}{dx^n} < 0 \longrightarrow (\text{Unstable})$$

Here n is even, $n = 2, 4, 6, \ldots\ldots$ (any even value).

⇒ If all even higher derivatives are zero then the point neither "stable" nor "Unstable."

Ex-1 $V(x) = \alpha x^4$ when $\alpha > 0$

Find equilibrium point and check it is stable or unstable ?

Sol^n Let x_0 is equilibrium point.

$$\left(\frac{dV(x)}{dx}\right)_{x=x_0} = 0$$

$$4\alpha x_0^3 = 0$$

$$\Rightarrow x_0 = 0$$

So x_0 is a equilibrium point.

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 12\alpha x_0^2$$

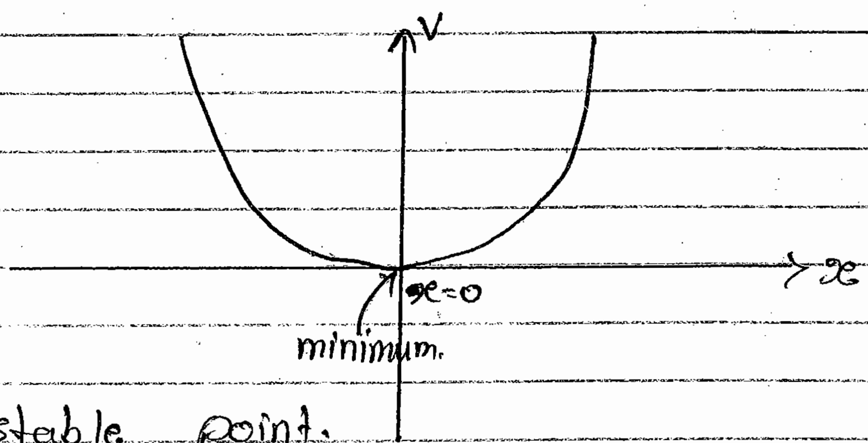
$$= 0 \quad \{\because x_0 = 0\}$$

ferther we check higher derivative.

$$\left.\frac{d^4V}{dx^4}\right|_{x=x_0} = 24\alpha > 0$$

So $x = x_0$ is stable point.

Graph of αx^4 :-

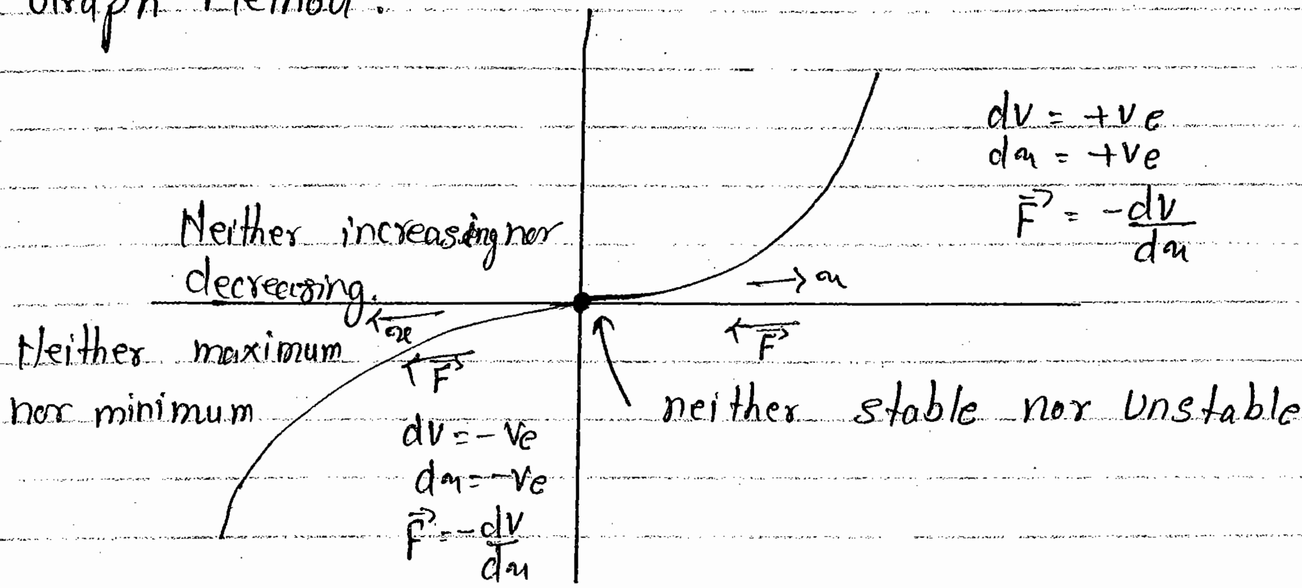


So,

$x = 0$ is a stable point.

Q. $V = \alpha x^3$, $x > 0$ find equilibrum point and check it is stable or unstable.

Sol^n Graph Method :-



Result :- Here for +ve displacement force is −ve but for −ve displacement it is also negative. So it is neither stable nor unstable.

Second Method :-

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 3\alpha x^2$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 6\alpha x$$

$$\left.\frac{d^3V}{dx^3}\right|_{x=x_0} = 6\alpha$$

$$\left.\frac{d^4V}{dx^4}\right|_{x=x_0} = 0$$

So $x = x_0$ neither stable point nor unstable

Q. If $V(x) = x(x-2)^2$, how many stable and unstable points are there?

Sol^n Condition of equilibrium (x_0) :-

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$$

$$\Rightarrow (x_0-2)^2 + 2x_0(x_0-2) = 0$$

$$\Rightarrow (x_0-2)\left[(x_0-2) + 2x_0\right]$$

$$\Rightarrow (x_0-2)\left[3x_0-2\right] = 0$$

$$\Rightarrow \left.\begin{array}{l} x_0 = 2 \\ x_0 = 2/3 \end{array}\right\} \text{Two equilibrium point.}$$

at $x_0 = 2$:-

$$\left.\frac{d^2V}{dx^2}\right|_{x_0=2} = 12 - 8 = 4 > 0$$

So $x_0 = 2$ is stable point.

$$\left.\frac{d^2V}{dx^2}\right|_{x_0=\frac{2}{3}} = 6 \times \frac{2}{3} - 8 = -4 < 0$$

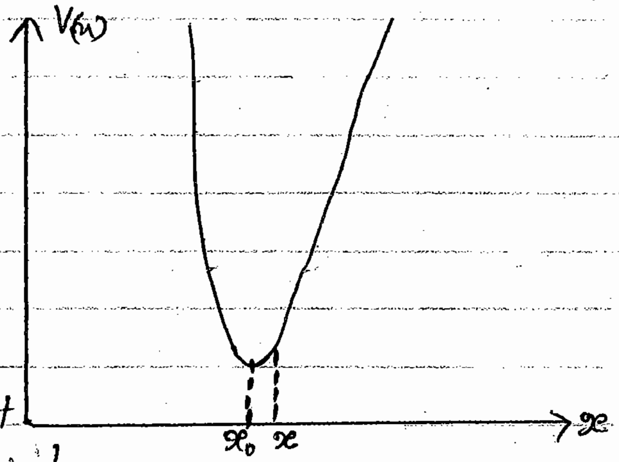
So $x_0 = \frac{2}{3}$ is a unstable point.

* **Frequency of Oscillation about stable equilibrium point {for small oscillation}:-**

We know a particle of mass m moving under potential $V(x) = V_0 + \frac{1}{2}kx^2$ has freq. of oscillation $\omega = \sqrt{\frac{k}{m}}$, about stable equilibrium point, then we can expand $V(x)$ about x_0 using Taylor's series expansion as,

$$V(x) = V(x_0) + (x-x_0)\left.\frac{dV}{dx}\right|_{x=x_0} + \frac{(x-x_0)^2}{2!}\left.\frac{d^2V}{dx^2}\right|_{x=x_0} + \frac{(x-x_0)^3}{3!}\left.\frac{d^3V}{dx^3}\right|_{x=x_0} + \cdots\cdots$$

Let a particle of mass m is moving under a potential $V(x)$. Let x_0 be a stable point.



In the above expansion- $(x-x_0)$ is a displacement from stable equilibrium point.

At the equilibrium position -

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$$

therefore, if $(x-x_0)$ be small then higher power terms are neglected.

$$V(x) = V(x_0) + \frac{(x-x_0)^2}{2!}\left.\frac{d^2V}{dx^2}\right|_{x=x_0}$$

it is greater than 0 so let it is $= k$.

$$\boxed{V(x) = V(x_0) + \frac{1}{2} K (x - x_0)^2}$$

So force :-

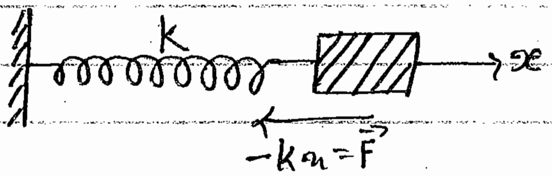
$$\vec{F} = -\frac{dV}{dx}$$

$$\vec{F} = -(x - x_0)K$$

$$\boxed{\vec{F} = -K(x - x_0)}$$

Here force is $(-K)$ times displacement and this is a case of S.H.M.

An example of S.H.M :-



So in the case of S.H.M.

$$\Rightarrow -Kx = ma$$

$$\Rightarrow \boxed{\ddot{x} = -\left(\frac{K}{m}\right) x}$$

$$\omega^2 = \frac{K}{m}$$

So $$\boxed{\omega = \sqrt{\frac{K}{m}}}$$

ω : Freq. of oscillation
m : mass of particle
K : Force Constant.

Here $$\boxed{K = \left.\frac{d^2V}{dx^2}\right|_{x=x_0}}$$ x_0 : stable point.

Here frequency of oscillation is independent of displacement so it is true for small oscillation ω is independent of amplitude. If oscillation is not small then ω is dependent of amplitude (energy).

Conclusions :-

$\Rightarrow$ $$\omega = \sqrt{\frac{k}{m}} \quad \text{where } k = \left.\frac{d^2V}{dx^2}\right|_{x=x_0}$$

x_0 is stable point.

$\Rightarrow$ It is not applicable in the case of -

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 0$$

$\Rightarrow$ If $\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 0$. Then we will write -

$$V(x) = V(x_0) + \frac{(x-x_0)^3}{\lfloor 3} \left.\frac{d^3V}{dx^3}\right|_{x=x_0}$$

$$V(x) = V(x_0) + 3\frac{(x-x_0)^2}{\lfloor 3} . k$$

$$\boxed{V(x) = V(x_0) - \frac{(x-x_0)^2}{\lfloor 2} . k}$$

$$\boxed{\vec{F} = -\frac{dV}{dx}}$$

$$\boxed{\vec{F} = -\frac{(x-x_0)^2}{\lfloor 2} . k}$$

Here $\vec{F} \propto (x-x_0)^2$ So it is not in S.H.M.

$\Rightarrow$ If $V = V(\theta)$ then -

$$\omega = \sqrt{\frac{K}{I}} \quad , \quad K = \left.\frac{d^2V}{d\theta^2}\right|_{x=\theta_0}$$

Where θ_0 = Stable point.

I = Moment of Inertia.

CSIR - 2012

A-4

Ques 9 Consider the motion of a classical particle in a one dimensional double-well potential $V(x) = \frac{1}{4}(x^2-2)^2$. If the particle is displaced infinitesimally from the minimum of the positive X-axis (and friction is neglected), then-

(a) the particle will execute simple harmonic motion in the right well with an angular freq. $\omega = \sqrt{2}$

(b) the particle will execute simple harmonic motion in the right with an angular frequency $\omega = 2$.

(c) the particle will switch between the right and left well. {Not possible in classical mechanics.}

(d) the particle will approach the bottom of the right well and settle there {Here no settle because there is no friction}.

Sol^n

$$V(x) = \frac{1}{4}(x^2-2)^2$$

$$\therefore \quad \omega = \sqrt{\frac{K}{m}} \quad , \quad K = \left.\frac{d^2V}{dx^2}\right|_{x=x_0}$$

At equilibrium point -

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$$

$$x_0(x_0^2-2) = 0$$

$$\Rightarrow \quad \left.\begin{array}{l} x_0 = 0 \\ x_0 = \pm\sqrt{2} \end{array}\right\} \text{ there is 3 equilibrium points.}$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 3x_0^2 - 2$$

at $x = x_0 = 0$

$$\frac{d^2V}{dx^2} = -2 < 0 \quad \left(\text{So } x_0 = 0 \text{ is unstable point}\right)$$

at $x = x_0 = \pm\sqrt{2}$

$$\left.\frac{d^2V}{dx^2}\right|_{x_0 = \pm\sqrt{2}} = 6 - 2 = 4 > 0 \left\{ \begin{array}{l} x_0 = \pm\sqrt{2} \text{ is} \\ \text{stable point} \end{array} \right.$$

So there are two stable points $(\pm\sqrt{2})$.

$$\therefore \boxed{k = 4}$$

$$\therefore \omega = \sqrt{\frac{4}{1}} \qquad \left\{ \begin{array}{c} \text{take } m = 1 \\ \text{Unit mass} \end{array} \right\}$$

Angular freq. $\boxed{\omega = 2}$

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* How to Plot a graph {V(x) v/s x} :-

Steps :-

(i) Put $V(x) = 0$ and find the value of x.

(ii) Put $dV/dx = 0$ and find the value of x.

(iii) Find position of maxima and minima i.e. check second derivative.
If second derivative is zero then check higher derivative.

(iv) Connect all the points where $V(x) = 0$ by making appropriate maxima or minima.

(V) Consider $x \longrightarrow \pm\infty$

Ex- $V(x) = \frac{1}{4}(x^2-2)^2$

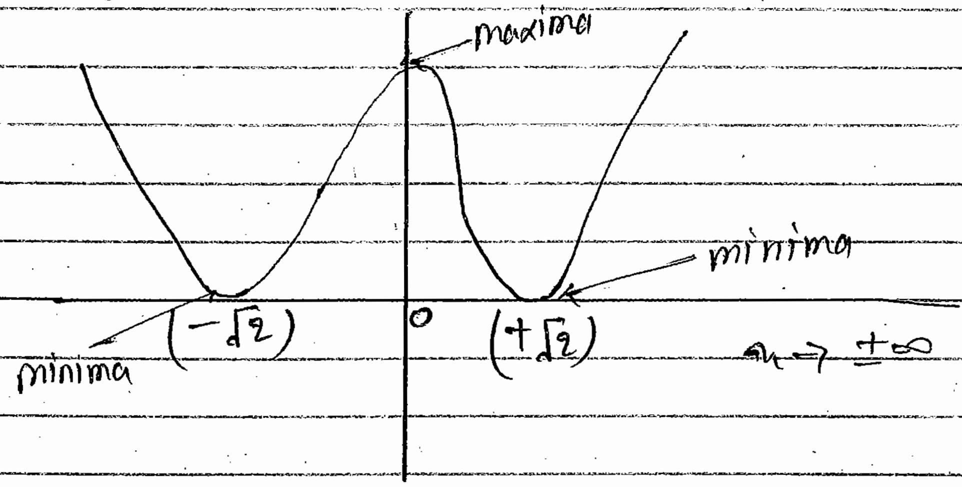
Sol^n $V(x) = 0 \Rightarrow (x^2-2)^2 = 0$

$x = \pm\sqrt{2}$

$\frac{dV}{dx} = 0$ find $x = 0, \pm\sqrt{2}$

$\frac{d^2V}{dx^2} > 0$ at $x = \pm\sqrt{2}$ i.e. minima

$\frac{d^2V}{dx^2} < 0$ at $x = 0$ $\}$ i.e. it is maxima



A-4
Q.14 A particle of mass 'm' is moving under potential $V(x) = ax^3 - bx$, frequency of oscillation about the stable equilibrium posion is -

(a) $\left[\frac{12ab}{m^2}\right]^{1/4}$ ✓ (b) $\left[\frac{6ab}{m^2}\right]^{1/4}$ (c) $\left[\frac{4ab}{m^2}\right]^{1/4}$ (d) $\left[\frac{3ab}{m^2}\right]^{1/4}$,

Sol^n

$V(x) = ax^3 - bx$

let x_0 is equilibrium poin then

$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 3ax_0^2 - b = 0$$

$$3ax_0^2 = b$$

$$x_0^2 = \frac{b}{3a}$$

$$\boxed{x_0 = \pm\sqrt{\frac{b}{3a}}}$$

Now $\frac{d^2V}{dx^2} = 6ax$

$= \sqrt{\frac{b}{3a}} \cdot 6a > 0$ stable (minima)

$= -6a\sqrt{\frac{b}{3a}} < 0$ Unstable (maxima).

$\therefore \left.\frac{d^2V}{dx^2}\right|_{x=x_0} = K = 6a\sqrt{\frac{b}{3a}} = \left[\frac{36a^2b}{3a}\right]^{1/2}$

So angular freq (ω) $= \sqrt{\frac{K}{m}}$ about stable point.

So $\omega = \sqrt{\left(\frac{\cancel{36}^{12}a^{\cancel{2}}b}{\cancel{3a}}\right)^{1/2} \cdot \frac{1}{m}}$

$$\omega = \sqrt{\left(\frac{12ab}{m^2}\right)^{1/2}}$$

$$\boxed{\omega = \left(\frac{12ab}{m^2}\right)^{1/4}}$$

Ans

Note :-

In a particle in which speed is to be calculated & Potential energy is given. Then we apply conservation of energy.

Ex- $V(x) = A - Bx + Cx^3$

For equilibrium point -

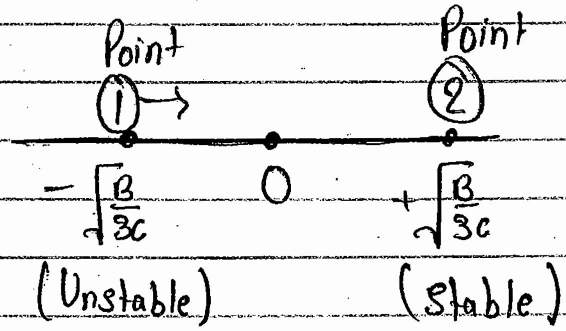
$$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$$

$$-B + 3Cx_0^2 = 0$$

$$\boxed{x_0 = \pm\sqrt{\frac{B}{3C}}}$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 6Cx_0$$

If system is conservative then conservation of energy.



$$(P.E. + K.E.)_{at\ point\ 1} = (P.E. + K.E.)_{at\ point\ 2}$$

$$\cancel{A} + B\sqrt{\frac{B}{3C}} - \frac{CB}{3C}\sqrt{\frac{B}{3C}} + 0 = \cancel{A} - B\sqrt{\frac{B}{3C}} + \frac{CB}{3C}\sqrt{\frac{B}{3C}} + \frac{1}{2}mv^2$$

$$2B\sqrt{\frac{B}{3C}} - \frac{2CB}{3C}\sqrt{\frac{B}{3C}} = \frac{1}{2}mv^2$$

$$\sqrt{\frac{B}{3C}}\left[2B - \frac{2\cancel{C}B}{3\cancel{C}}\right] = \frac{1}{2}mv^2$$

$$2B\sqrt{\frac{B}{3C}}\left[1-\frac{B}{3}\right] = \frac{1}{2}mv^2$$

$$\boxed{\left\{\frac{4B}{m}\sqrt{\frac{B}{3C}}\left[1-\frac{B}{3}\right]\right\}^{1/2} = V}$$

Ans

June 2013

Q. There is a particle of mass m moving with potential $V(x) = ax - bx^3$ initially particle is at rest at stable point what is minimum speed should be given to it. so that its motion becomes unstable.

Sol^n

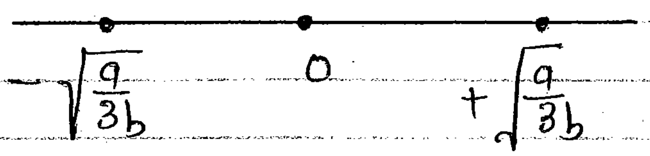
$$V(x) = ax - bx^3$$

$$\left.\frac{dV(x)}{dx}\right|_{x_0} = a - 3bx_0^2 = 0$$

$$\Rightarrow 3bx_0^2 = a$$

$$\Rightarrow x_0 = \pm\sqrt{\frac{a}{3b}}$$

So that equilibrium point is, $+\sqrt{\frac{a}{3b}}$ and $-\sqrt{\frac{a}{3b}}$



$$\frac{d^2V(x)}{dx^2} = -6bx_0 < 0$$

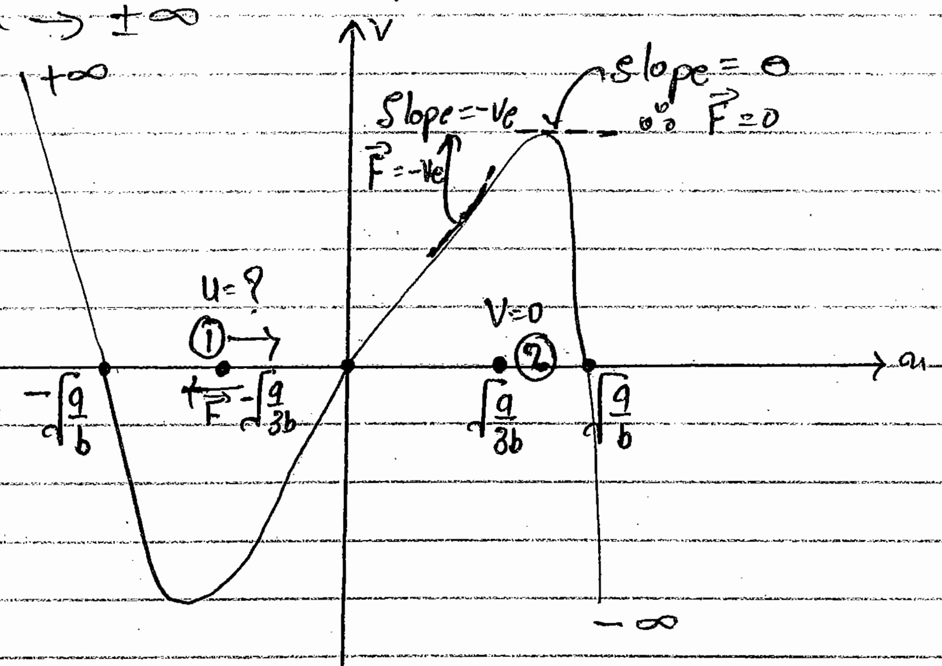
So $-\sqrt{\frac{a}{3b}}$ is stable point (min^m potential point.

$V(x) = 0$

$\Rightarrow x(a-bx^2) = 0$

$$x = 0, \quad x = \pm\sqrt{\frac{a}{b}}$$

and $x \to \pm\infty$



for minimum speed that particle reaches unstable point and stop there.

Applying Energy Conservation at point ① and ②:

$$(P.E. + K.E.)_{I} = (P.E. + K.E.)_{II}$$

$$\Rightarrow \frac{1}{2}mu^2 - a\sqrt{\frac{a}{3b}} + b\left(\frac{a}{3b}\right)^{3/2} = 0 + a\sqrt{\frac{a}{3b}} - b\left(\frac{a}{3b}\right)^{3/2}$$

$$\Rightarrow \frac{1}{2}mu^2 = 2a\sqrt{\frac{a}{3b}} - 2b\left(\frac{a}{3b}\right)^{3/2}$$

$$\frac{1}{2} mu^2 = 2\sqrt{\frac{a}{3b}}\left\{a - \cancel{b}\left(\frac{a}{3\cancel{b}}\right)\right]$$

$$mu^2 = 4\sqrt{\frac{a}{3b}}\left\{\frac{3a-a}{3}\right\} = 4\sqrt{\frac{a}{3b}}\left\{\frac{2a}{3}\right\}$$

$$mu^2 = \sqrt{\frac{a}{3b}}\left\{\frac{8a}{3}\right\} = \left\{\frac{64a^2.a}{27b}\right\}^{1/2}$$

$$u^2 = \left\{\frac{64a^3}{27m^2b}\right\}^{1/2}$$

$$\boxed{u = \left\{\frac{64a^3}{27m^2b}\right\}^{1/4}}$$

Q.6 A particle of mass $m=4$ moves along the x-axis under the influence of the potential $V(x) = 2(e^{-\frac{3x}{2}} - 2e^{-\frac{3}{4}x})$, if the particle oscillates with small amplitude around the minimum potential, what is the period of the oscillation.

(a) 0.12 (b) 1.33 (c) 8.37 ✓ (d) 11.17

Sol^n

$$V(x) = 2\left[e^{-\frac{3}{2}x} - 2e^{-\frac{3}{4}x}\right]$$

$$\omega = \sqrt{\frac{k}{m}}, \quad k = \left.\frac{d^2V}{dx^2}\right|_{x=x_0} \text{ (stable)}$$

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$$

$$2\left[\frac{-3}{2}e^{-\frac{3}{2}x_0} + \frac{3}{2}e^{-\frac{3}{4}x_0}\right] = 0$$

$$e^{-\frac{3}{2}x_0} = e^{-\frac{3}{4}x_0}$$

Taking ln on both side.

$$\cancel{-}\frac{\cancel{3}}{2}x_0 = \cancel{-}\frac{\cancel{3}}{4}x_0$$

$$\left(\frac{1}{2}-\frac{1}{4}\right)x_0 = 0$$

$$\because \left(\frac{1}{2}-\frac{1}{4}\right) \neq 0$$

$$\therefore \boxed{x_0 = 0}$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 2\left[\frac{9}{4}e^{-\frac{3}{2}x_0} - \frac{9}{8}e^{-\frac{3}{4}x_0}\right]$$

$$= 2\left[\frac{9}{4}-\frac{9}{8}\right]$$

$$= 2\times\frac{9}{8}$$

$$= \frac{9}{4} = k = +ve$$

So $x_0 = 0$ is stable point.

$$\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$$\frac{2\times 3.14}{T} = \sqrt{\frac{9}{4\times 4}}$$

$$\frac{2\times 3.14}{T} = \frac{3}{4}$$

$$T = \frac{8\times 3.14}{3} = \frac{25.12}{3}$$

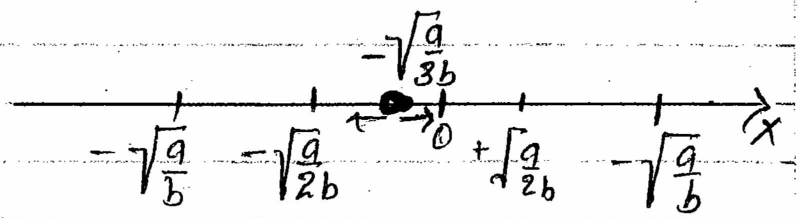
$$\boxed{T = 8.3733}$$

Q.7 Force acting on a particle along x-axis is given to be $F(x) = 2bx^3 - ax$. If the particle is released from point $x = -\sqrt{\frac{a}{3b}}$ it will move towards which of the following points.

(a) 0 ✓ (b) $+\sqrt{\frac{a}{3b}}$ (c) $-\sqrt{\frac{a}{2b}}$ (d) $-\sqrt{\frac{a}{b}}$

Sol^n

$\vec{F}(x) = 2bx^3 - ax$



$$f(x) = -2\cancel{b} \cdot \frac{a}{3\cancel{b}}\sqrt{\frac{a}{3b}} + a\sqrt{\frac{a}{3b}}$$

$$= \left(\frac{-2a}{b} + a\right)\sqrt{\frac{a}{3b}}$$

$$= \frac{a}{3b}\sqrt{\frac{a}{3b}} \quad (+ve)$$

So it move in the direction of 0.

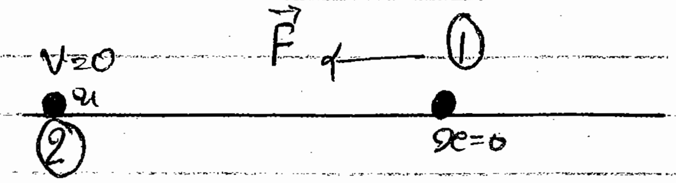
Q.12 A particle of mass 2kg is released at $x = 0$m. The particle is acted upon by a force whose potential is expressed as $V(x) = 2x - 4x^3$ joule. The force on the particle at the point where it will again come to rest for the first time is –

(a) -4 Newton (b) $+4$ Newton ✓ (c) zero Newton

(d) $+2$ Newton.

Sol^n

$V(x) = 2x - 4x^3$



$\vec{F} = -\frac{dV}{dx} = 12x^2 - 2$

Put $x = 0$

$F = -2$ so F is $-ve$

Apply conservation of energy at ① & ②

$$(K.E. + P.E.)_I = (K.E. + P.E.)_{II}$$

$$(0+0) = (0 + 2x - 4x^3) \quad \begin{pmatrix}\text{Put } x=0 \\ \text{in } V(x)\end{pmatrix}$$

$$x = 0, \pm\frac{1}{\sqrt{2}}$$

$-\frac{1}{\sqrt{2}}$ is the point where particle is in rest for first time.

So $\vec{F} = 12x^2 - 2 = 12\left(-\frac{1}{\sqrt{2}}\right)^2 - 2$

$= 6 - 2 = 4$ Newton. Ans

A-4

Q.10 The total energy E of the particle of mass m executing small oscillations about the origin along on the X-direction is given by, $E = \frac{1}{2}mv^2 + V_0 \cosh\left(\frac{X}{L}\right)$, where V_0 and L are positive constants. The time period T of oscillation is -

(a) $T = \frac{1}{2\pi}\sqrt{\frac{m}{V_0}}$ (b) $T = 2\pi\sqrt{\frac{L}{m}}$ (c) $T = \pi L\sqrt{\frac{m}{E}}$

(d) $T = 2\pi\sqrt{\frac{mL^2}{V_0}}$

Sol^n

$$E = \underbrace{\frac{1}{2}mv^2}_{K.E.} + \underbrace{V_0 \cosh\left(\frac{x}{L}\right)}_{P.E.}$$

$$V(x) = V_0 \cosh\left(\frac{x}{L}\right)$$

∵ Origin is stable point (given)

$x = 0$

$$\omega = \sqrt{\frac{k}{m}}$$

$$k = \left.\frac{d^2V}{dx^2}\right|_{x=0} = \frac{V_0}{L^2}\left(\cosh\frac{x}{L}\right)_{x_0}$$

$$\boxed{K = \frac{V_0}{L^2}}$$

$$\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{V_0}{L^2 m}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{V_0}{L^2 m}}$$

$$\boxed{T = 2\pi\sqrt{\frac{L^2 m}{V_0}}}$$ Ans

Q.11 A particle of mass 'm' is moving under a one dimensional potential $V(x) = ax^3 - bx$. Due to the force acting on the particle its kinetic energy changes as the particle moves from one point to the other. What can be maximum change in K.E. of the particle in this case -

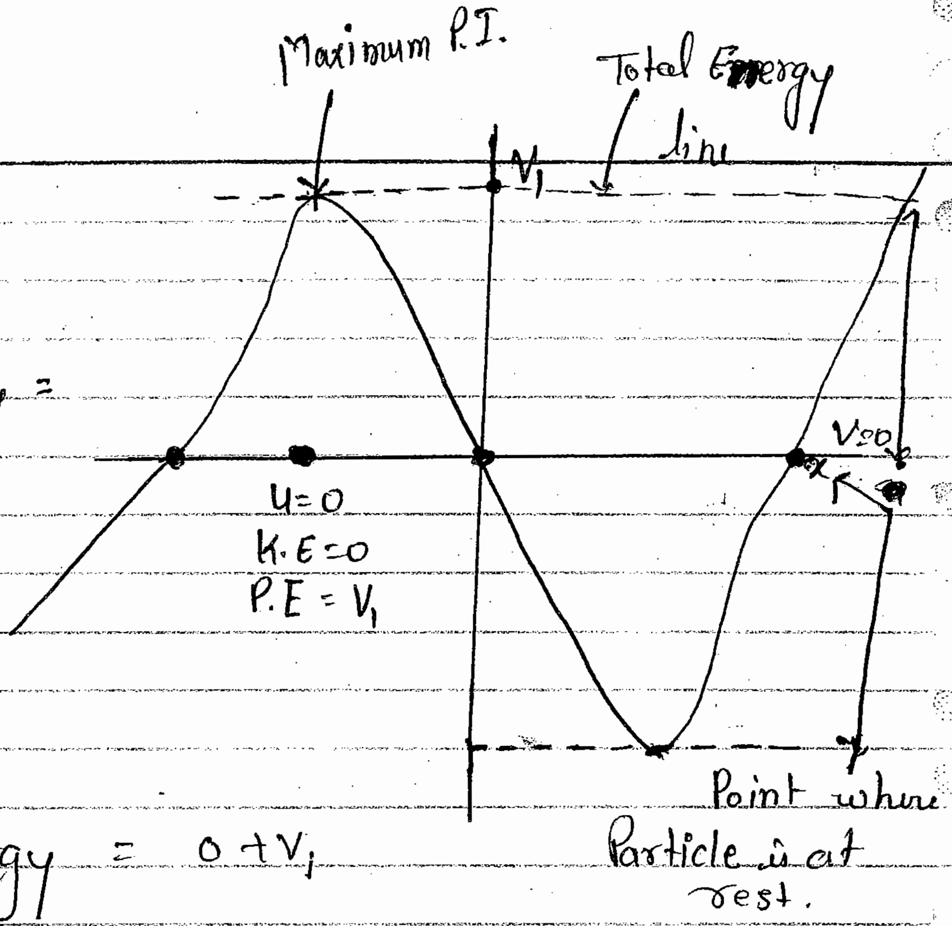
(a) $\sqrt{\frac{16b^3}{3a}}$ (b) $\sqrt{\frac{2b^3}{3a}}$ (c) $\sqrt{\frac{4b^3}{3a}}$ (d) $\sqrt{\frac{8b^3}{3a}}$

Sol^n

$V(x) = ax^3 - bx$

$\because$ Total Energy =

K.E + P.E.



So

Total Energy $= 0 + V_1$

Total Energy $= V_1$

Maximum Change in K.E. = Max^m change in P.E.

$= V_{max} - V_{min}$

Q.14 A particle of mass 'm' is moving under potential $V(x) = ax^3 - bx$, frequency of oscillation about the stable equilibrium position is -

(a) $\left[\frac{12ab}{m^2}\right]^{1/4}$ (b) $\left[\frac{6ab}{m^2}\right]^{1/4}$ (c) $\left[\frac{9ab}{m^2}\right]^{1/4}$ (d) $\left[\frac{3ab}{m^2}\right]^{1/4}$

Sol^n $V(x) = ax^3 - bx \Rightarrow \left.\frac{dV}{dx}\right|_{x=x_0} = 3ax_0^2 - b$

$3ax_0^2 - b = 0 \Rightarrow 3ax_0^2 = b \Rightarrow x_0^2 = \frac{b}{3a}$ $\boxed{x_0 \pm \sqrt{\frac{b}{3a}}}$

So $\pm\sqrt{\frac{b}{3a}}$ are two equilibrium point.

$\left.\frac{d^2V}{dx^2}\right|_{x_0=\sqrt{\frac{b}{3a}}} = 6ax_0 = 6a\sqrt{\frac{b}{3a}} > 0$

So $x_0 = \sqrt{\frac{b}{3a}}$ is minima or stable point.

$\left.\frac{d^2V}{dx^2}\right|_{x_0=-\sqrt{\frac{b}{3a}}} = 6ax_0 = -6a\sqrt{\frac{b}{3a}} < 0$

So $x_0 = -\sqrt{\frac{b}{3a}}$ is maxima or unstable point.

$\because$ $K = \left.\frac{d^2V}{dx^2}\right|_{x=x_0}$ where x_0 is stable point

So $K = 6a\sqrt{\frac{b}{3a}} = \cancel{36}\sqrt{\frac{a^2b}{\cancel{3a}}} = \sqrt{\frac{36a^2b}{3a}}$

$= \sqrt{\frac{36ab}{3}}$

$\because$ $\omega = \sqrt{\frac{K}{m}} = \sqrt{\left(\frac{36ab}{3m^2}\right)^{1/2}}$

$\omega = \sqrt{\left(\frac{12ab}{m^2}\right)^{1/2}} = \left(\frac{12ab}{m^2}\right)^{1/4}$

$$\boxed{\omega = \left(\frac{12ab}{m^2}\right)^{1/4}}$$ Ans

15 In the previous question force on the particle is maximum at

(a) $x = 0$ (b) $x = \sqrt{\frac{b}{a}}$ (c) $x = \sqrt{\frac{b}{3a}}$ (d) $x = -\sqrt{\frac{b}{3a}}$

Sol^n $V(x) = ax^3 - bx$

$$\vec{F} = -\frac{dV}{dx} = -(3ax^2 - b) = b - 3ax^2$$

for $\vec{F}$ to be max^m or min^m

$$\frac{dF}{dx} = 0$$

$$-6ax = 0$$

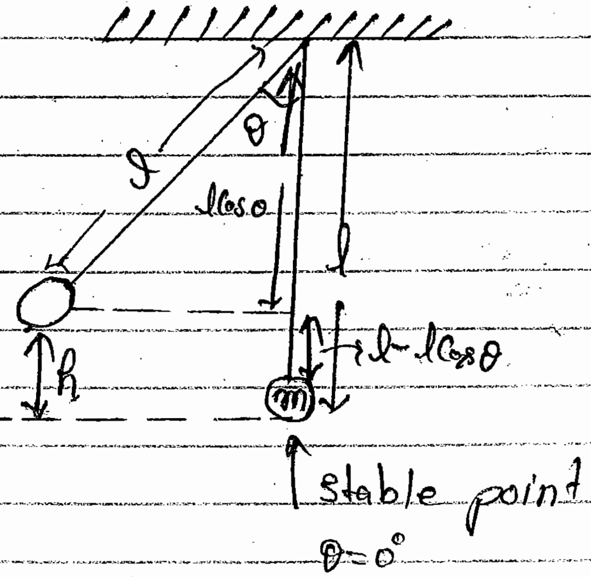
$$\therefore -6a \neq 0$$

$$\therefore \boxed{x = 0}$$

$$\frac{d^2F}{dx^2} = -6a \quad (\text{min}^m)$$

$$\therefore F_{max} = b - 0 = b \quad \text{at} \ \boxed{x = 0}$$

* Time Period of Simple Pendulum :-



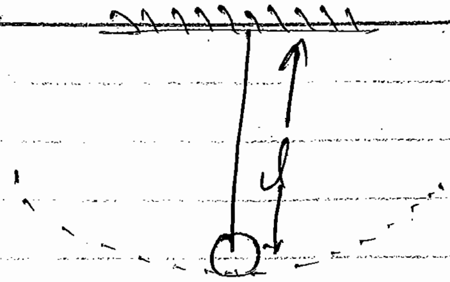
$$P.E = mgh$$

$$\boxed{P.E = mgl(1 - \cos\theta)}$$

$$K = \left.\frac{d^2V}{d\theta^2}\right|_{\theta=\theta_0}$$

$$= mgl \cos\theta \Big|_{\theta=0}$$

$$\boxed{K = mgl}$$



$$I = \sum_i m_i r_i^2 = ml^2$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{\cancel{m}g\cancel{l}}{\cancel{m}l^{\cancel{2}}}}$$

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

Note :-

⇒ K in Linear Case :-

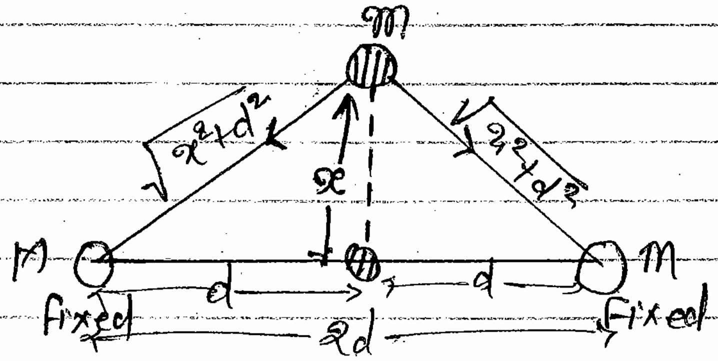
$$K = \left.\frac{d^2V}{dx^2}\right|_{x=0} \qquad \omega = \sqrt{\frac{K}{m}}$$

⇒ K in Angular Case :-

$$K = \left.\frac{d^2V}{d\theta^2}\right|_{\theta=\theta_0} \qquad \omega = \sqrt{\frac{K}{I}}$$

Ques | If m is slightly displaced along perpendicular bisector, what is its frequency of oscillation (Use P.I. method.)

Sol^n



P.E of the system $(V_x) = -\dfrac{2GMm}{\sqrt{x^2+d^2}} - \dfrac{GM^2}{2d}$

Stable point is $x=0$

$$K = \left.\frac{d^2V}{dx^2}\right|_{x=0}$$

$$\frac{dV}{dx} = -2GMm\frac{d}{dx}(x^2+d^2)^{-1/2}$$

$$= \frac{2GMm\,x}{(x^2+d^2)^{3/2}}$$

$$\frac{d^2V}{dx^2} = 2GMm\frac{d}{dx}\frac{x}{(x^2+d^2)^{3/2}}$$

$$= 2GMm\left[\frac{1.(x^2+d^2)^{3/2} - x.\frac{3}{2}\times(x^2+d^2)^{1/2}.2x}{(x^2+d^2)^3}\right]$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=0} = \frac{2GMm\cancel{d^3}}{d^{\cancel{6}}} = \frac{2GMm}{d^3}$$

$$\omega = \sqrt{\frac{k}{m}}$$

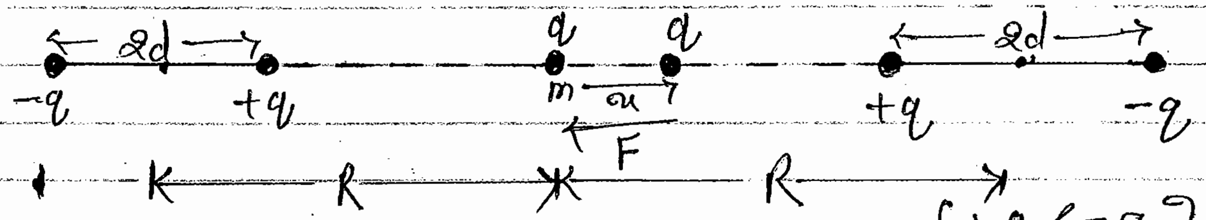
$$\omega = \sqrt{\frac{2GM\cancel{m}}{d^3\cancel{m}}} = \sqrt{\frac{2GM}{d^3}}$$

Angular frequency $\boxed{\omega = \sqrt{\frac{2GM}{d^3}}}$ Ans

CSIR - 2013 (June)

Q. If particle of mass m and charge +q is slightly displaced from given position what will be its frequency of oscillation ($d \lll R$).

Soln



$$V(x) = \frac{1}{4\pi\epsilon_0} Q.q \left[\frac{1}{(R-d+x)} - \frac{1}{(R+d+x)} + \frac{1}{R-d-x} - \frac{1}{R+d-x}\right]$$

{+q & -q are fixed}

$$\frac{d^2V}{dx^2} = \frac{Qq}{4\pi\epsilon_0}\left[\frac{2}{(R-d+x)^3} - \frac{2}{(R+d+x)^3} + \frac{2}{(R-d-x)^3} - \frac{2}{(R+d-x)^3}\right]$$

$\because$ $x = 0$ is stable point -

$$\left.\frac{d^2V}{dx^2}\right|_{x=0} = \left[\frac{qQq}{4\pi\epsilon_0}\left\{\frac{1}{(R-d)^3} - \frac{1}{(R+d)^3}\right\}\right]$$

$\because$ $d <<< R$

$$\text{so } K = \frac{qQq}{4\pi\epsilon_0 R^3}\left[\left(1-\frac{d}{R}\right)^{-3} + \left(1+\frac{d}{R}\right)^{-3}\right]$$

$$= \frac{\cancel{q}Qq}{\cancel{4}\pi\epsilon_0 R^3}\left[\cancel{1}+\frac{3d}{R} - \cancel{1} + \frac{3d}{R}\right]$$

$$= \frac{6Qqd}{\pi\epsilon_0 R^4}$$

$$\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{6dqQ}{\pi\epsilon_0 R^4}}$$

$$\boxed{\omega = \sqrt{\frac{6dQq}{\pi\epsilon_0 R^4}}}$$ Ans

06|Aug|2014

Non-Inertial Frames of Reference And Pseudo Forces

* Non-Inertial Frames :-

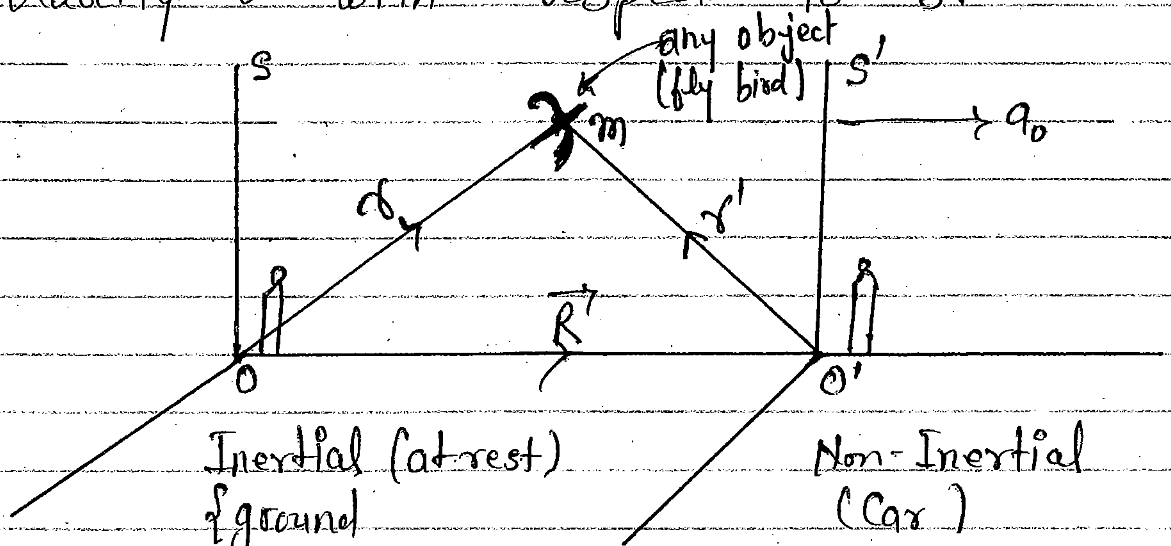
An accelarated frame is called Non-Inertial frames.

Accelarated frames are of two types-

1. Linearly Accelarated frame
2. Uniformly Rotating frame (ω = constt) OR Rotating frame.

1 Linearly Accelarated Frame :-

Let us consider two frame S and S' where S is at rest and S' is moving with constant velocity 'v' with respect to S.



Let a_0 = accelaration of S'.

$$\vec{R} + \vec{r'} = \vec{r}$$

$$\vec{r'} = \vec{r} - \vec{R} \quad \text{--- (1)}$$

Differentiate twice with respect to time -

$$\frac{d^2\vec{r'}}{dt^2} = \frac{d^2\vec{r}}{dt^2} - \frac{d^2\vec{R}}{dt^2}$$

$$\vec{a'} = \vec{a} - \vec{a_0}$$

both side multiplying by m :-

$$m\vec{a'} = m\vec{a} - m\vec{a_0}$$

$$\boxed{\vec{F'} = \vec{F} - m\vec{a_0}}$$ Pseudo Force.

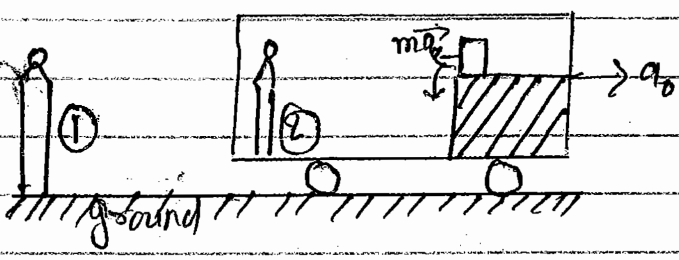
force on object at seen from s'

force on object at seen from s

$$\boxed{\vec{F'} = \vec{F} + (-m\vec{a_0})}$$

Note :-

Cause of fall of body in the transparent bus :-



Explanation of :-

first Person :- (Which is at ground) :-

Cause of fall of body in bus is ; There is no sufficient friction.

Second Person :-

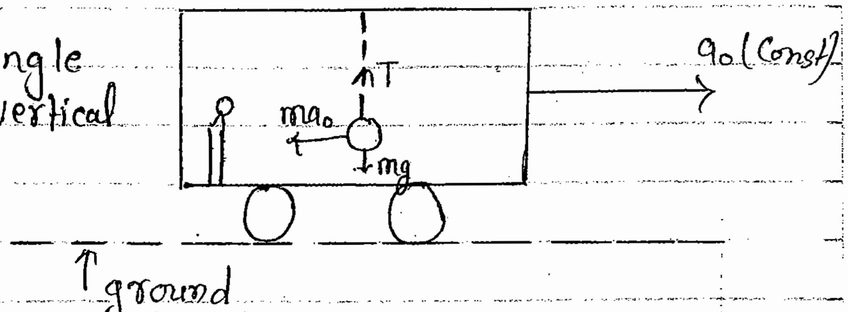
Due to pseudo force.

⇒ The origin of pseudo force is not known.

⇒ Concept of pseudo force is used when observation is made from non-inertial frame.

* Simple Pendulum in linearly accelerated frame :-

Let string makes angle θ with downward vertical in equilibrium position.



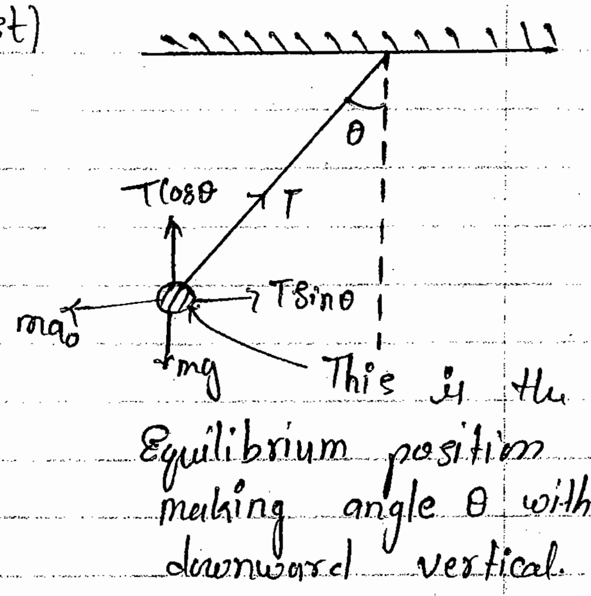
$T\sin\theta = ma_0$ —(I)

$T\cos\theta = mg$ —(II)

(I) / (II)

$\tan\theta = \frac{a_0}{g}$

$$\boxed{\theta = \tan^{-1}\frac{a_0}{g}}$$

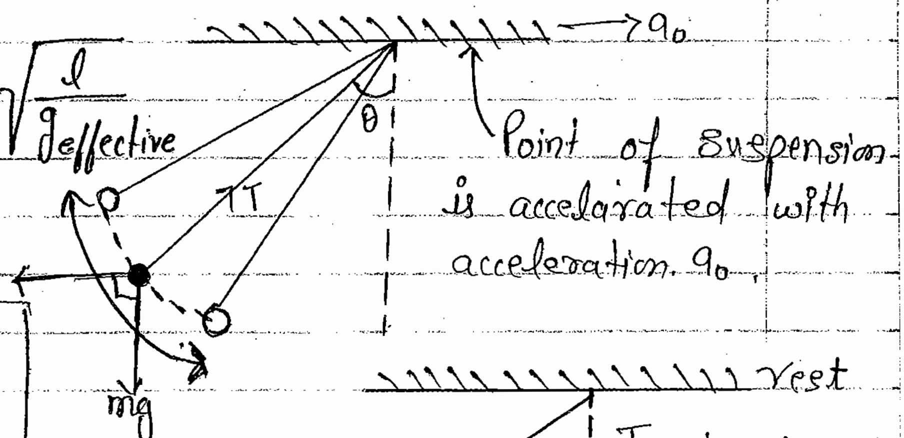


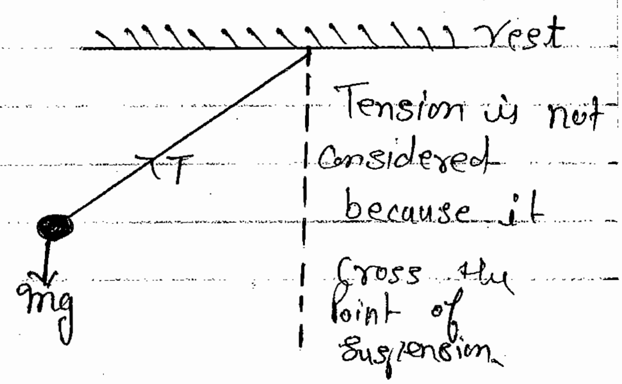
* Time Period of oscillation in accelarated frame :-

Time Period $(T) = 2\pi\sqrt{\frac{l}{g_{effective}}}$

$\therefore g_{eff} = \sqrt{g^2 + a_0^2}$

$$\therefore \boxed{T = 2\pi\sqrt{\frac{l}{\sqrt{g^2+a_0^2}}}}$$

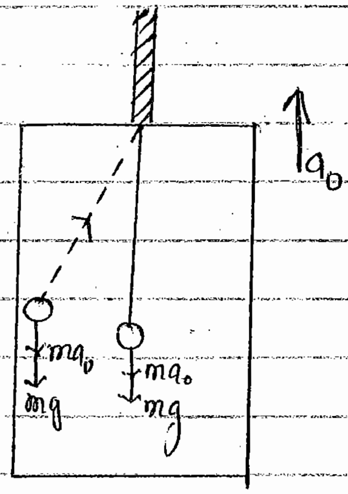




* Pendulum in a Lift :-

In this case :-

$$g_{eff} = a_0 + g$$

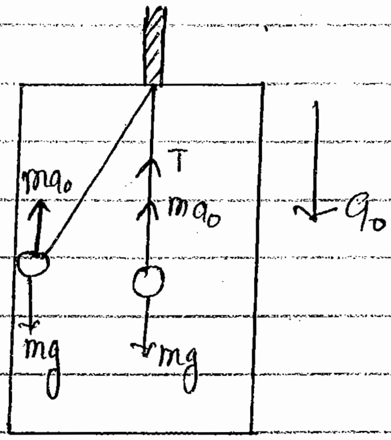


$$\therefore\ T = 2\pi\sqrt{\frac{l}{g_{effective}}}$$

So

$$\boxed{T = 2\pi\sqrt{\frac{l}{g + a_0}}}$$

In this case :-

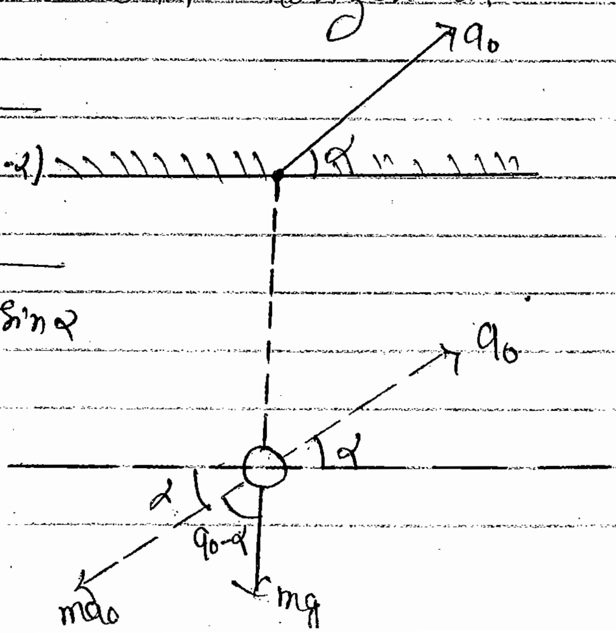


$$g_{eff} = g - a_0$$

$$a_0 < g$$

$$\boxed{T = 2\pi\sqrt{\frac{l}{g - a_0}}}$$

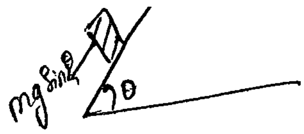
* When point of suspension is accelarated at an angle α ~~ver~~ with horizontal :-



$$g_{eff} = \sqrt{g^2 + a_0^2 + 2ga_0\cos(90-\alpha)}$$

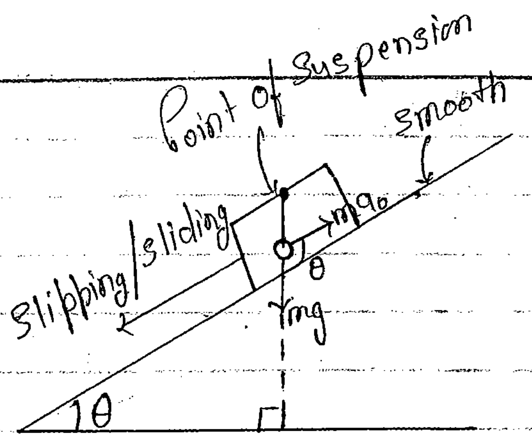
$$g_{eff} = \sqrt{g^2 + a_0^2 + 2ga_0\sin\alpha}$$

$$\boxed{T = 2\pi\sqrt{\frac{l}{g_{eff}}}}$$



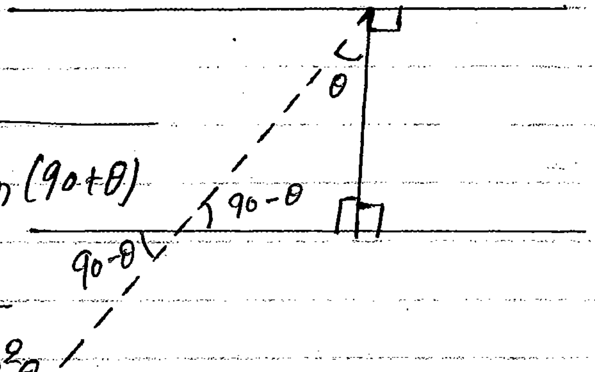
$v = mg\sin\theta \quad a = \frac{\cancel{m}g\sin\theta}{\cancel{m}} \Rightarrow \boxed{a = g\sin\theta}$

Q. What is the time period of pendulum.



$a_0 = g\sin\theta$

$\therefore \quad \theta = 90 + \theta$



$g_{eff} = \sqrt{g^2 + a_0^2 + 2ga_0\sin(90+\theta)}$

$= \sqrt{g^2 + (g\sin\theta)^2 - 2g(g\sin\theta)\sin(90+\theta)}$

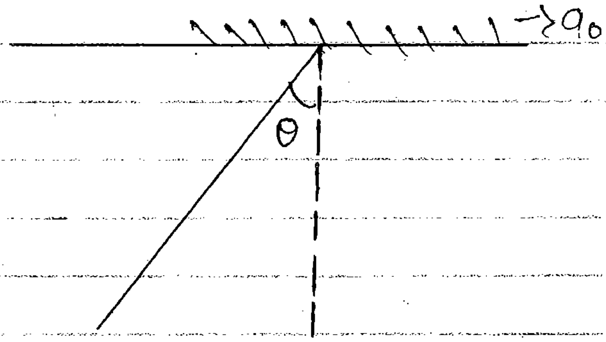
$= \sqrt{g^2 + g^2\sin^2\theta - 2g^2\sin^2\theta}$

$= g\sqrt{1 + \sin^2\theta - 2\sin^2\theta} \quad = g\sqrt{1 - \sin^2\theta}$

$g_{eff} = g\cos\theta$

So $\boxed{T = 2\pi\sqrt{\frac{l}{g\cos\theta}}}$

Q. Acceleration of point of suspension is $\sqrt{3}g$ in horizontal direction.



Sol^n

$\theta = \tan^{-1}\left(\frac{a_0}{g}\right)$

$= \tan^{-1}(\sqrt{3})$

$\boxed{\theta = 60^\circ}$ Ans

Given -

In stationary case time period $= T$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

then

$$T' = 2\pi\sqrt{\frac{l}{g_{eff}}}$$

$$g_{eff} = \sqrt{a_0^2 + g^2}$$

$$= \sqrt{3g^2 + g^2} = \sqrt{4g^2}$$

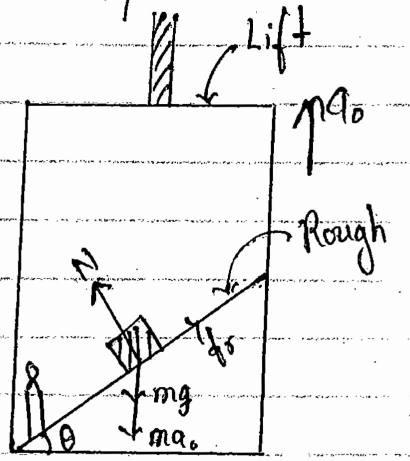
$$g_{eff} = 2g$$

$$\boxed{T' = 2\pi\sqrt{\frac{l}{2g}}}$$

$$\boxed{T' = \frac{T}{\sqrt{2}}}$$ Ans

13/08/2014

Q. Block is not sliding on inclined plane what work done by the friction on Block. During the time lift move up by distance 'h'



Sol^n As seen from inside the lift, block is at rest.

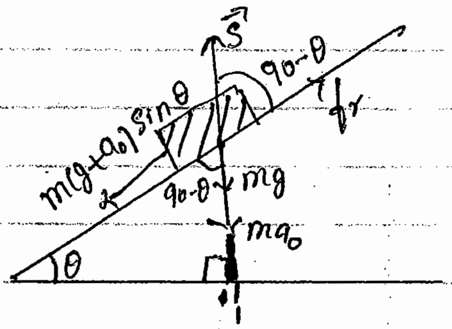
Therefore -

$$f_r = m(g+a_0)\sin\theta$$

Work done = Force . Displacement

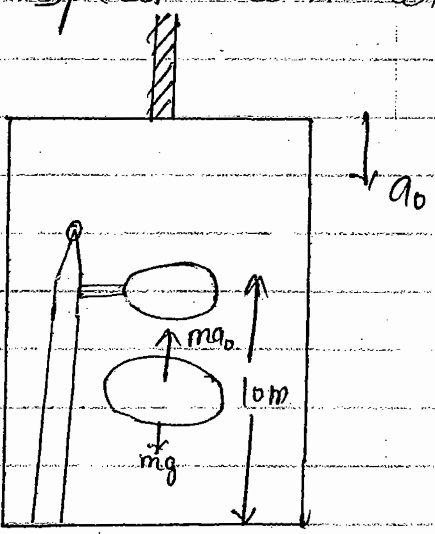
$$= \vec{F}.\vec{S}$$

$$= f_r h \cos(90-\theta)$$

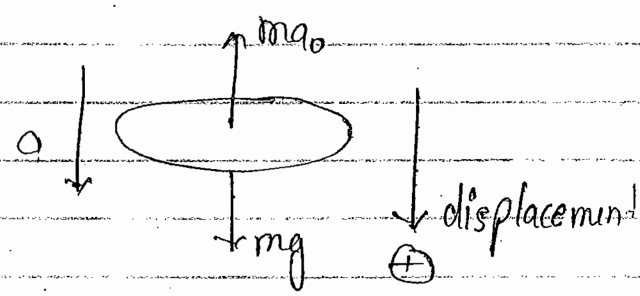


$$\boxed{W.D. = m(g+a_0)h\sin^2\theta}$$

Q. A lift is going downward direction with $a_0 = 4\,m/s^2$. A person drops a coin at 10 m distance from initial speed 0 with initial velocity. With what speed coin will hit the floor of lift.



Sol^n



Let a = acc^n of coin as seen by person -

$$mg - ma_0 = ma$$

$$a = (g - a_0)$$

$$v^2 = u^2 + 2as$$

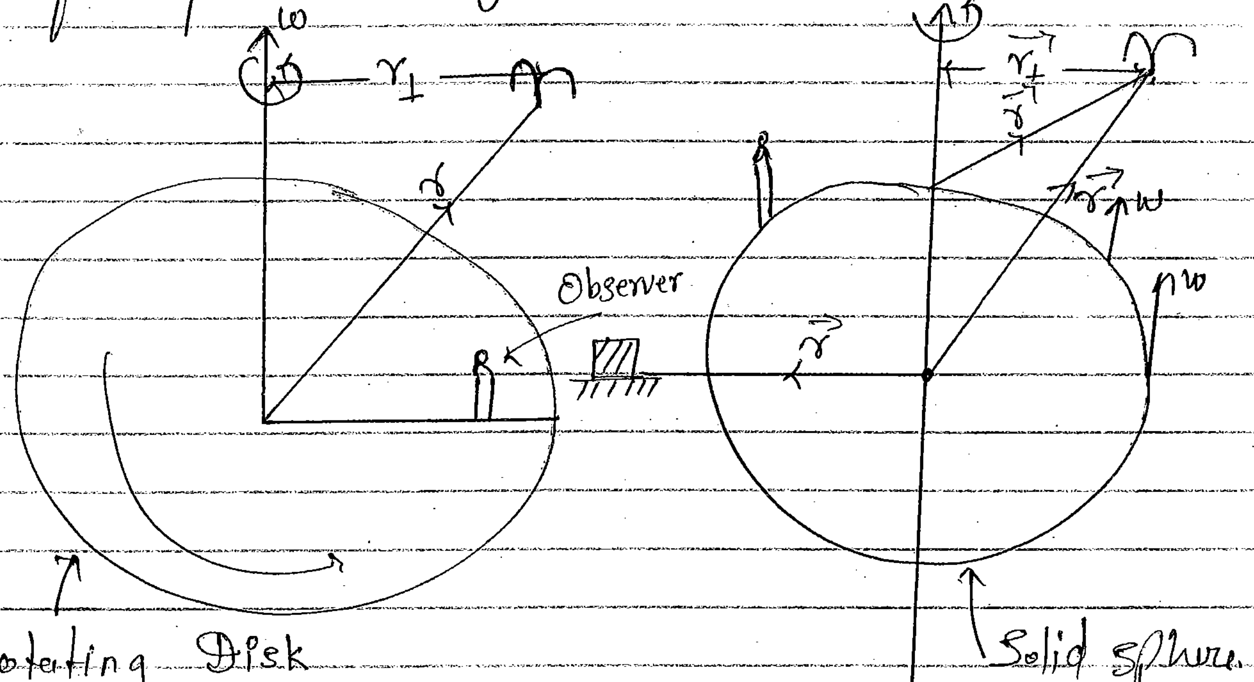
$$v^2 = 0 + 2(g - a_0) \times 10$$

$$= 2 \times 6 \times 10$$

$$v^2 = 120$$

$$\boxed{v = \sqrt{120}\ m/sec}$$ Ans

* Uniformly Rotating frame :- { ω = Constt } :-

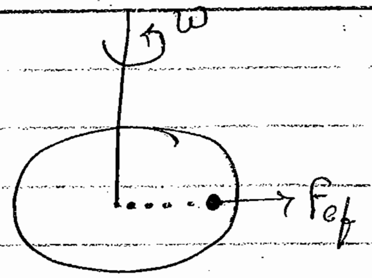


Rotating frame is example of Non-Inertial frame.

When observation is made from rotating frame on all objects (irrespective of their location). following pseudo forces appears to act on them.

1. Centrifugal force :-

$$\boxed{F_{cf} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})}$$



m = mass of object.
$\vec{\omega}$ = angular velocity of rotating frame
$\vec{r}$ = position vector of object (with respect to a stationary point of frame).

Magnitude of Centrifugal Force :-

$$\boxed{\vec{F}_{cf} = m\omega^2 r_\perp}$$

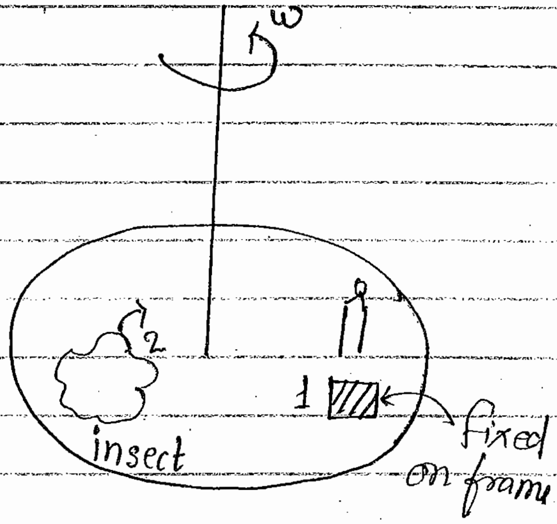
$r_\perp$ = $\perp^{or}$ distance of object from axis of rotation.

⇒ Centrifugal force acts on all object irrespective of their state of motion.

2. Coriolis Force :-

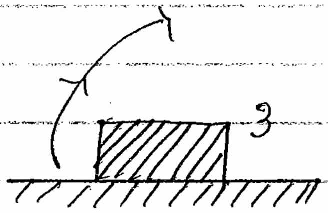
Coriolis :- A name of Mechanical Engineer
Coriolis was a mechanical engineer. He is observing force on the liquid which is in rotating arm of the machine and unfortunately he discover a different type of force which is after his name Kla coriolis force.

"This force is appears to act on all object which appear to be moving when seen from the rotating frame (or object have speed with respect to rotating frame.)"



Here Coriolis force act on object 2 & 3 but not on 1.

Expression of Coriolis force :-



$$\vec{F}_{cor.} = -2m(\vec{\omega} \times \vec{v}_r)$$

v_r = Velocity of object with respect to rotating frame.

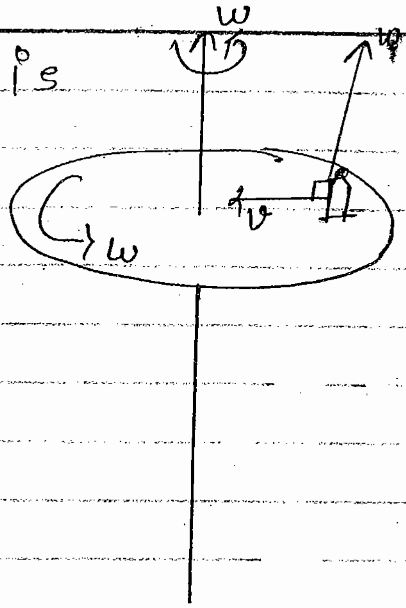
$$\boxed{\vec{F}_{cor} = 2m(\vec{v}_r \times \vec{\omega})}$$

* Some Basic Problems Based on Coriolis force :-

A-5

Q.22 A circular platform is rotating with a uniform angular speed ω counterclockwise about an axis passing through its center and perpendicular to its plane as shown in the figure. A person of mass m walks radially inwards with a uniform sppeed v on the plateform. The magnitude and the direction of the coriollis force (with respect to the direction along which the person walks is ?

Curl the finger v to ω to get direction of



Axis is parallel to plane of paper.

$V_r = v$

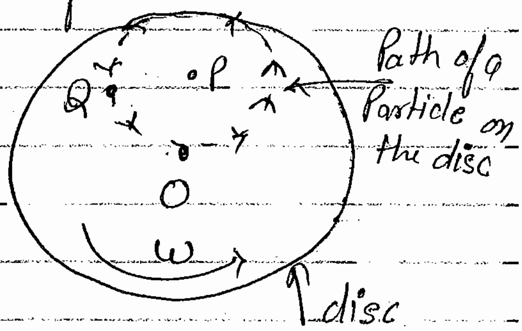
$|\vec{F}_{cor}| = 2mV_r\omega \sin 90°$

$= 2mv\omega \quad \{\sin 90° = 1\}$

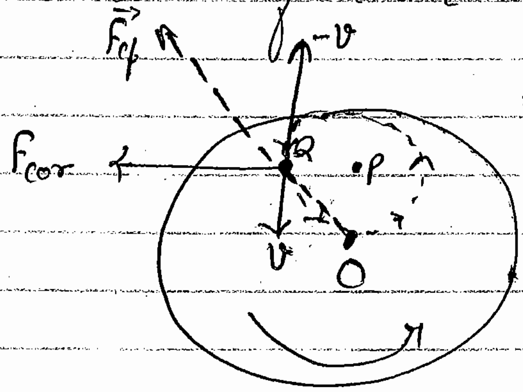
$\boxed{F_{cor} = 2mv\omega}$ towards his Right

So option (C) is Correct

Q-5 Q.4 A circular disc is rotating in anticlockwise sence as shown in the figure. On the disc a particle moves in anticlockwise circle with center at P. At the instant particle at Q, which of the following options correctly represents directions of centrifugal and coriolis forces ['O' is center of the disc].



Sol



Axis is ⊥ar to the Plane of Paper.

When particle (insect) is at Q show direction of centrifugal and Coriolis force.

"Direction of centrifugal force ($\vec{F}_{cf}$) is always perpendicular to axis from object of rotation and away from it."

So option (d) is correct.

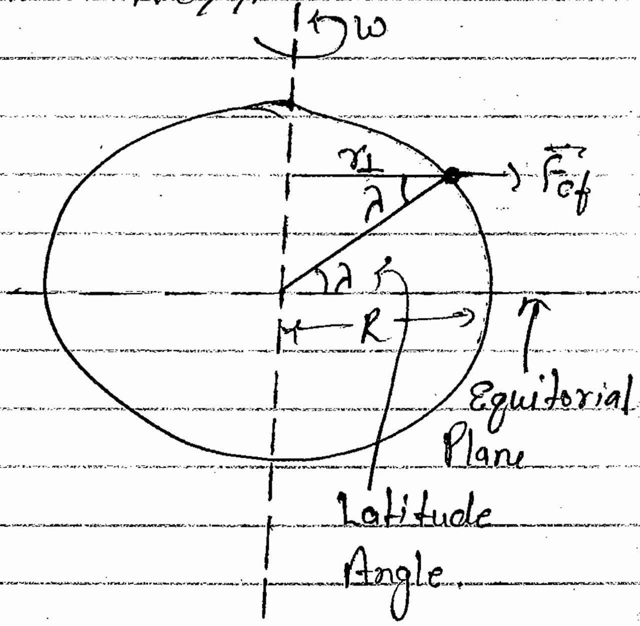
A-5

Q.5 A particle of mass m is lying on earth's surface at a location where latitude is λ. If ω be angular velocity of earth's spining motion and R be the radius of the earth then, centrifugal force on the particle is - ?

(a) $m\omega^2 R \sin\lambda$ (b) $m\omega^2 R \cos\lambda$ ✓

(c) $2m\omega^2 R \sin\lambda$ (d) $2m\omega^2 R \cos\lambda$

Sol



$\vec{F}_{cf} = m\omega^2 r_\perp$

$\cos\lambda = \dfrac{r_\perp}{R}$

$r_\perp = R\cos\lambda$

So $\boxed{\vec{F}_{cf} = m\omega^2 R\cos\lambda}$

At equiator (at $\lambda = 0$) :-

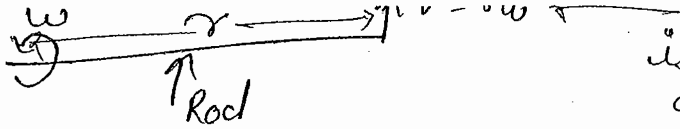
$\boxed{\vec{F}_{cf} = m\omega^2 R = max^m}$

At poles (At $\lambda = 90^\circ$)

$\boxed{\vec{F}_{cf} = 0 = min^m}$

Imp.

Q. A disc is rotating in horizontal plane with uniform angular velocity ω an object on mass m is lying on the ground in the same plane at a distance r from the center what is centrifugal and

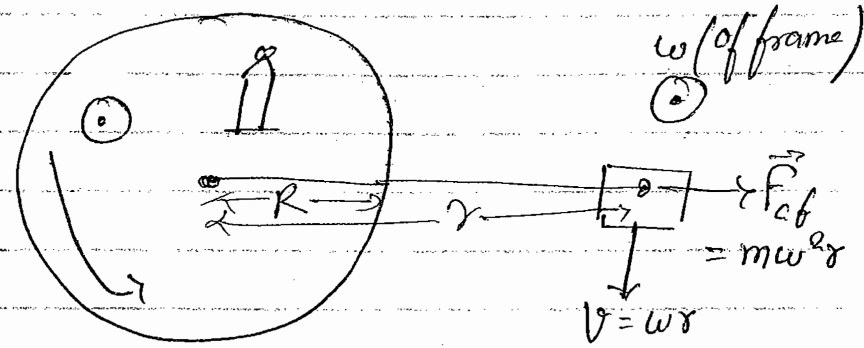


This velocity is due to rotation of rod.

Coriolis force on the object as seen by a person, standing on the disc.

Sol^n

$v = \omega r$ → Velocity arising out of rotation is given by $\omega r_{\perp}$



$$\vec{F}_{cor} = 2m\cancel{v_r}\cdot \omega \sin 90^\circ$$

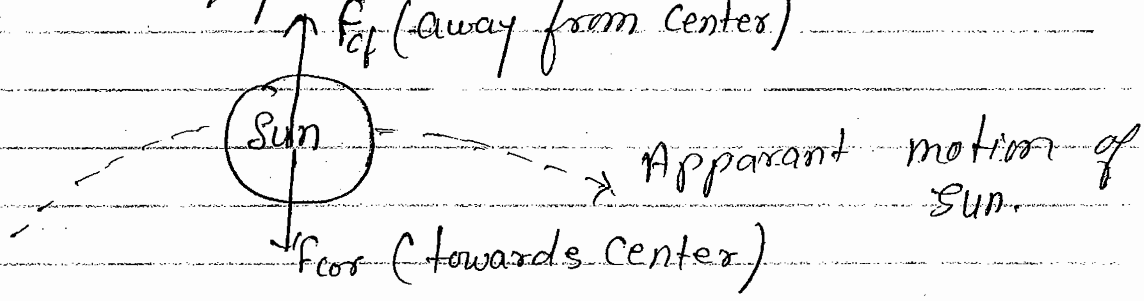
$$= 2mv_r \cdot \omega \cdot 1$$

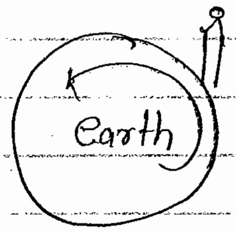
$$= 2m\omega r \cdot \omega$$

$$\boxed{\vec{F}_{cor} = 2m\omega^2 r}$$ Towards center.

$$\boxed{\vec{F}_{cf} = m\omega^2 r}$$ (away from center)

* Earth Sun System :-

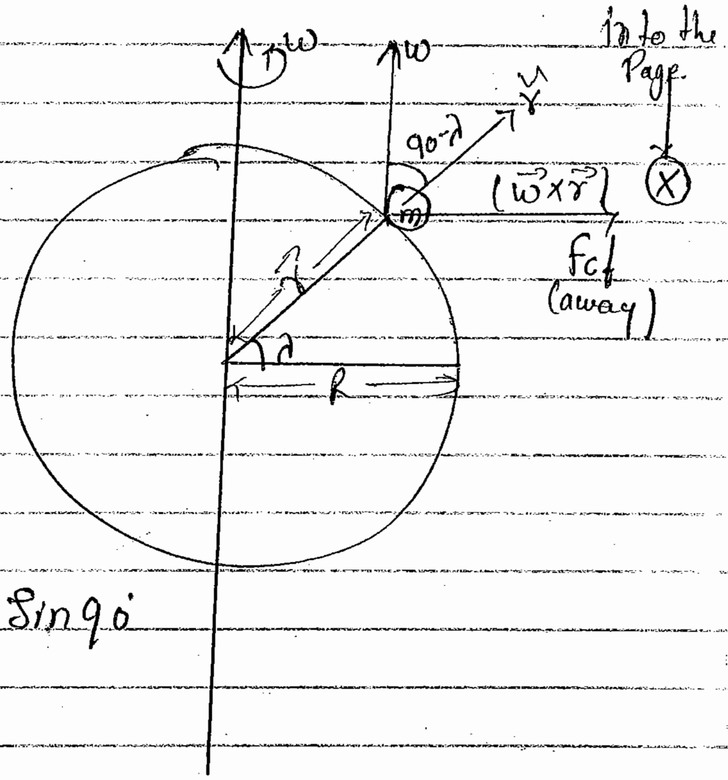




A-5

Q.5

2nd Second Method :-



$$\vec{F}_{cf} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\vec{F}_{cf} = m(\vec{\omega}\times\vec{r})\times\vec{\omega}$$

$$|\vec{F}_{cf}| = |m(\underline{\vec{\omega}\times\vec{r}})\,\underline{\times\vec{\omega}}|$$

$$\vec{F}_{cf} = m|\vec{\omega}\times\vec{r}|\,|\vec{\omega}|\sin 90^\circ$$

$$= m|\vec{\omega}\times\vec{r}|\cdot\omega\cdot 1$$

$$= m\omega|r|\sin(90-\lambda)\cdot\omega\cdot 1$$

$$\boxed{|\vec{F}_{cf}| = m\omega^2 R\cos\lambda}$$

Ans

* Earth: A non-Inertial (rotating) frame:

Earth spins about its axis, therefore, therefore it is a non-inertial frame.
Angular velocity of earth's spining motion is
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{24h} = 7.27\times10^{-5}\ rad/sec.$$
Due to rotation of earth following effects arise.

(a) All objects on earth's surface experience centrifugal force which is directed away from the axis of rotation.

(b) All objects moving on earth's surface experience coriolis force in addition to centrifugal force.

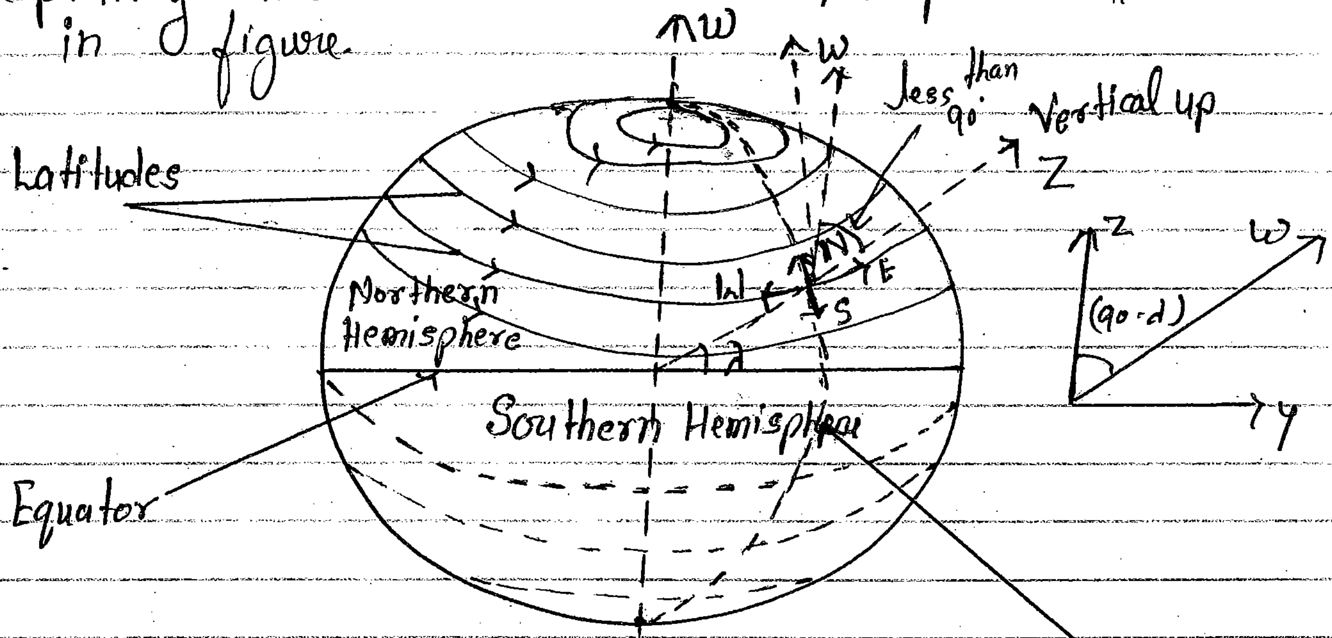
(c) An observer standing on earth's surface notices that on all objects a centrifugal and a coriolis force (if $v' \neq 0$) act irrespective of the location of the object. ~~This~~ That is the object which is being observed by the observer may or may not be on the earth's surface. For example we on the earth, see that the ~~Sun~~ Sun rises and sets. If we analyse the dynamics of the sun we will find that Sun experiences a centrifugal force away from the earth and a coriolis force towards the earth.

Coordinate System on Earth's Surface:-

To analyse dynamics of objects from earth's surface we define local cartesian co-ordinate system as follows-

(i) +X-axis : Along local east, −X axis : Along local west
(ii) +Y-axis : Along local north, −Y axis : Along local south
(iii) +Z-axis : Along local vertical upward
−Z-axis : Along local vertical downward.

Therefore angular velocity vector of earth's spining motion lies in Y-Z plane as shown in figure.



A point in northern Hemisphere. Longitude

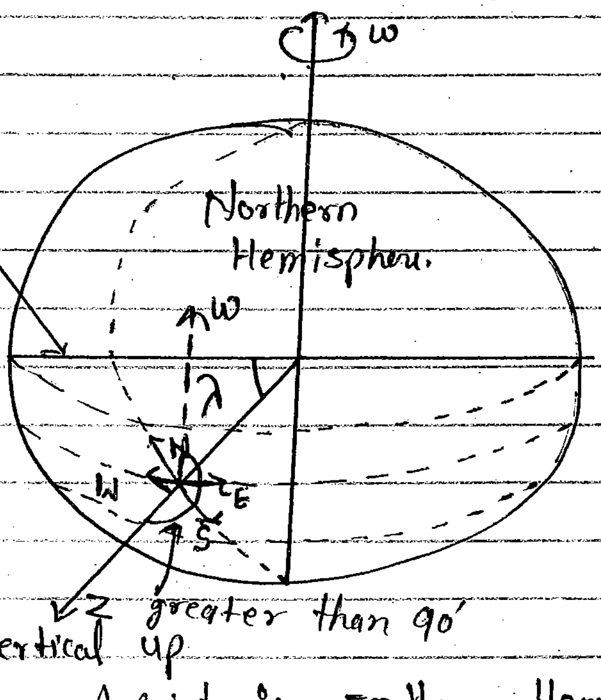
Therefore, at any point in northern hemisphere,

$\vec{\omega} = \omega(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$ Equator

When λ is latitude of the point.

And at any point in southern hemisphere.

$\vec{\omega} = \omega(\cos\lambda\,\hat{j} - \sin\lambda\,\hat{k})$



A point in southern Hemisphere.

A-5

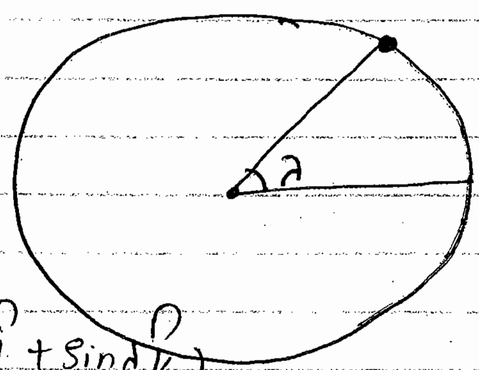
Q.6 In the previous question if the particle starts moving along the latitude from east to west with constant speed V_0, coriolis force on the particle will be (magnitude).

(a) $2m\omega V_0$ (b) $2mV_0\omega\cos\lambda$

(c) $2mV_0\omega\sin\lambda$ (d) $2mV_0\omega\sin\lambda\cos\lambda$

Soln.

$$\vec{\omega} = \omega(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$$



$$\vec{v} = -V_0\,\hat{i}$$

$$\vec{F}_{cor} = 2m\vec{v}\times\vec{\omega}$$
$$= -2mV_0\omega\,\hat{i}\times(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$$

$$\vec{F}_{cor} = -2mV_0\omega(\cos\lambda\,\hat{k} - \sin\lambda\,\hat{j})$$

$$|\vec{F}_{cor}| = 2mV_0\omega\sqrt{\cos^2\lambda + \sin^2\lambda}$$

$$\boxed{|\vec{F}_{cor}| = 2mV_0\omega}$$

* Eastward deviation of a freely falling Object in northern hemisphere:-

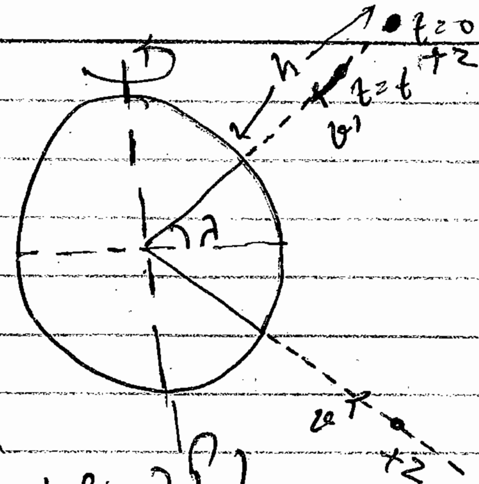
Suppose a particle falls from a height h at a place where latitude is λ.

Here we assume that $h \ll R_e$

Velocity of particle at time t is -

$$\vec{v}' = -gt\,\hat{k}$$

Coriolis force on the particle -



$$\vec{\omega} = \omega(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$$

So
$$\vec{F}_{cor} = 2m\vec{v}\times\vec{\omega}$$
$$= -2m\omega g t\hat{k} \times (\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$$
$$\vec{F}_{cor} = 2m\omega g\cos\lambda\, t\,\hat{i}$$

Force is towards east so particle will be deviate towards east, in both hemisphere.

Equation of motion of the particle in eastward direction is -

$$\vec{F}_{cor} = \frac{md^2x}{dt^3} \quad \text{or} \quad \frac{d^2x}{dt^2} = 2\omega g\cos\lambda t$$

We have following initial condition which we use to solve above equation -

at, $t=0$, $x=0, y=0, z=h$ —— (a)

at, $t=0$, $\frac{dx}{dt}=0$, $\frac{dy}{dt}=0$, $\frac{dz}{dt}=0$ —— (b)

Integrating the above differential equation we get, $\frac{dx}{dt} = \omega g\cos\lambda t^2 + c_1$

Using (b) we get $c_1 = 0$. Therefore $\frac{dx}{dt} = \omega g\cos\lambda t^2$

Integrating again we get, $x = \frac{1}{3}\omega g\cos\lambda\, t^3 + c_2$

Using (a) we get $c_2 = 0$

$$\therefore \quad \boxed{x = \frac{1}{3}\omega g \cos\lambda\, t^3}$$

Where t is total time taken to reach earth surface.

$$\therefore \quad t = \sqrt{\frac{2h}{g}}$$

$$\therefore \quad \boxed{x = \frac{1}{3}\omega g \cos\lambda \left(\frac{2h}{g}\right)^{3/2}}$$

If a particle falls from a height of 100 metre at equator ($\lambda = 0$) then using above equation we get.

$$x = \frac{1}{3} \times 7.27 \times 10^{-5} \times 9.8 \times \left(\frac{2 \times 100}{9.8}\right)^{3/2}$$

$$= 2.19 \approx 2.2 \text{ c.m.}$$

Note :-

The expression derived for eastward deviation is an approximate one. In the derivation we have assumed that particle is uniformly accelerated in downward direction which is not true, when particle attains a velocity in X (each direction) there arises a component of coriolis force in vertical direction, due to which downward acceleration becomes a function of V_x'. Because of V_x' component of velocity, particle also experiences a force in southward direction. Therefore a freely falling object in northern hemisphere actually deviates in south-east direction.

$E \to \hat{\varphi}$; $E \to \hat{x}$, $W \to -\hat{x}$ $\quad V = V1$

$N \to -\hat{\theta}$; $N \to \hat{y}$, $S \to -\hat{y}$

$S \to \hat{\theta}$

21/Aug/2014

* Coriolis and Centrifugal forces :-

$$\vec{F}_{cor} = 2m\,\vec{v}_r \times \vec{\omega} \qquad \text{Earth} \quad \omega = \frac{2\pi}{T}$$

$$\vec{F}_{cf} = m\omega^2 r_\perp \qquad = \frac{2\pi}{24}$$

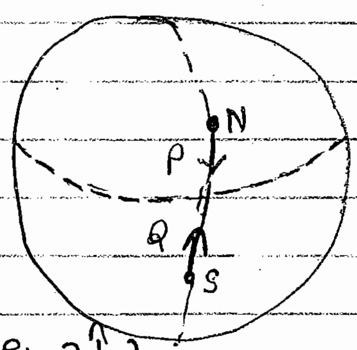
Since $\omega = 7.25 \times 10^{-5}$ rad/sec $\qquad \omega = 7.25 \times 10^{-5}$ rad/sec

In the Earth's case

$$|\vec{F}_{cf}| <<< |\vec{F}_{cor}|$$

So in the case of daviation we can not consider the Centrifugal force.

A-5
Q.20



Sol^n for P :-

$$v = v(-\hat{j})$$

$$\vec{F}_{cor} = 2m\,\vec{v} \times \vec{\omega}$$

$$= 2m\,v\omega\,(-\hat{j}) \times (\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$$

$$\vec{F} = 2mv\omega\sin\lambda\,(-i) \qquad \left\{ \because \omega = \omega(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k}) \right.$$

So force is in $-i$ direction or $-x$ direction

So daviation in $-x$ direction. or West direction.

For Q :-

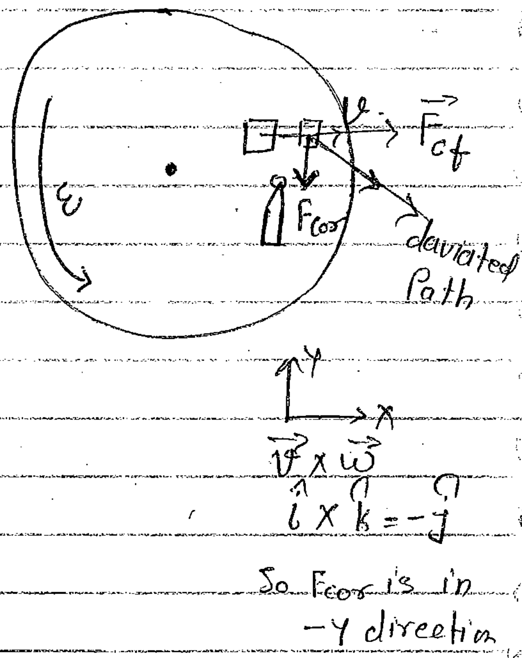
$$\vec{v} = v\,\hat{j}$$

$$\vec{F}_{cor} = 2mv\omega\, \hat{j} \times (\cos\lambda\, \hat{j} - \sin\lambda\, \hat{k})$$

$$\vec{F}_{cor} = 2mv\omega \sin\lambda\, (-\hat{i}) \quad \{\text{West direction}\}$$

So both P & Q deviate in West direction.

Q.21

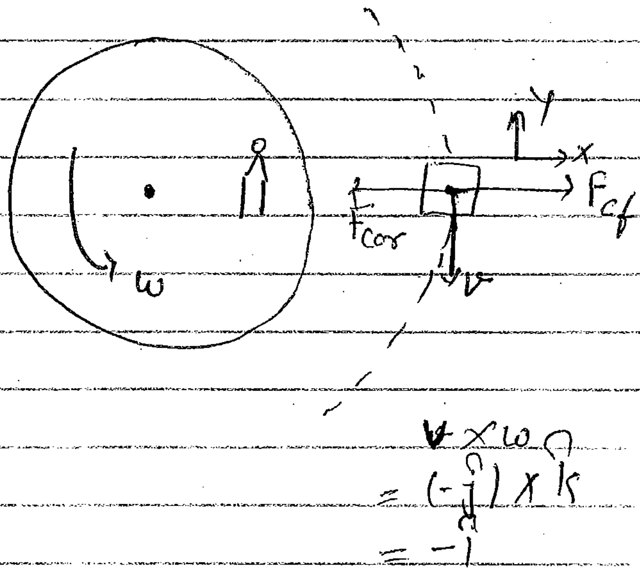


$$\vec{F}_{cf} = m\omega^2 r_\perp$$

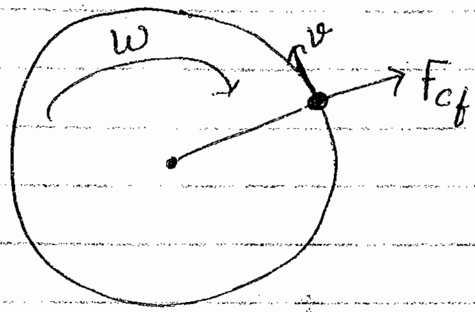
option ①.

Q.1 :-

(c)



Q.2



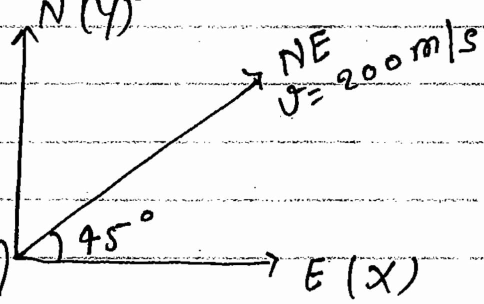
Direction of centrifugal force is independent of velocity

$$\vec{F}_{cf} = m\omega^2 r_\perp$$
$$= m\omega^2 R$$

Q.11

Sol^n

$\lambda = 45°$ N , $m = 2$ kg



$$\vec{F}_{cor} = 2m\,\vec{v}\times\vec{\omega}$$

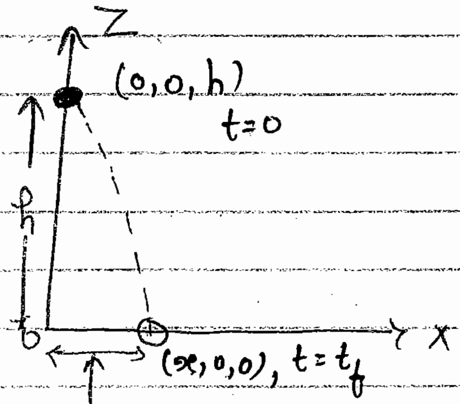
0.05 $$= 2\times 2\{200(\cos 45\,\hat{i} + \sin 45\,\hat{j}) \times 7.25\times 10^{-5}(\cos 45\,\hat{j} + \sin 45\,\hat{k})$$

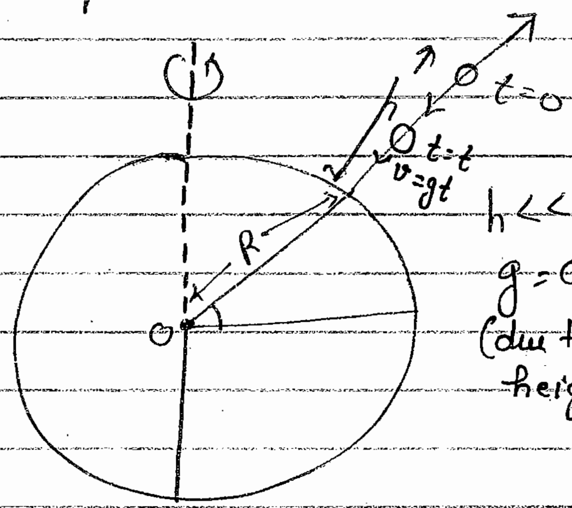
$$= 4\{200(\tfrac{1}{\sqrt{2}}\hat{i} + \tfrac{1}{\sqrt{2}}\hat{j}) \times 7.25\times 10^{-5}(\tfrac{1}{\sqrt{2}}\hat{j} + \tfrac{1}{\sqrt{2}}\hat{k})\}$$

$$\vec{v} = 200(\cos 45\,\hat{i} + \sin 45\,\hat{j})$$
$$\vec{\omega} = \omega(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$$
$$\vec{\omega} = 7.25\times 10^{-5}(\cos 45\,\hat{j} + \sin 45\,\hat{k})$$

=

* Value of East Ward deviation of an object falling on Earth Surface :-





h <<< R

g = Constt. (due to small height)

Eastward daviation or distance coverd in East direction.

$$\vec{F}_{cor} = 2mv(-\hat{k}) \times \omega(\cos\lambda\,\hat{j} \pm \sin\lambda\,\hat{k})$$

$$\boxed{\vec{F}_{cor} = 2mv\cos\lambda\,\hat{i}}$$

So the direction of force is $\hat{i}$ or $+x$ direc^n. or in East direction.

Accelaration in East direction -

$$a_x = \frac{F_x}{m} = \frac{2\cancel{m}\,v\omega\cos\lambda}{\cancel{m}}$$

$$a_x = 2v\omega\cos\lambda$$

$$\boxed{a_x = 2gt\omega\cos\lambda} \neq \text{Constt.}$$

Velocity at t = t

$v = u + gt$

$v = 0 + gt$

$v = gt.$

∵ Acceleration is not constt. so we use defination.

$$a_x = \frac{dv_x}{dt}$$

$$\frac{dv_x}{dt} = 2gt\omega \cos\lambda$$

$$v_x = 2g\omega\cos\lambda \cdot \frac{t^2}{2}$$

$$\frac{dx}{dt} = g\omega t^2 \cos\lambda$$

$$\int_0^x dx = g\omega\cos\lambda \int_0^t t^2\,dt$$

$$\boxed{x = \frac{1}{3} g\omega\cos\lambda\, t_f^3}$$

$\{ t_f =$ time of fall $\}$

⇒ To write eastward deviation in terms of height, Consider vertical motion :-

$$s = ut + \frac{1}{2}at^2$$

$$h = 0 + \frac{1}{2} g t_f^2$$

$$\boxed{t_f = \sqrt{\frac{2h}{g}}}$$

$u = 0,\ t = 0$

$a = g =$ Constt.

$t = t_f$.

$$\boxed{x = \frac{1}{3} g\omega\cos\lambda \left(\frac{2h}{g}\right)^{3/2}}$$

→ Memorise.

This is the final expression for eastward deviation.

At equator $\lambda = 0$

If $h = 100$ meter

So $x = 2.2$ cm.

 ← Helix

Eastward deviation is maximum at equator and minimum at poles {∵ at poles $\theta = 90^\circ$ and $\cos 90^\circ = 0$ }.

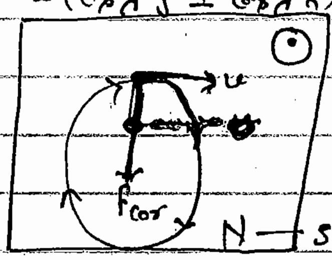
* Direction of deviation of horizontally thrown object on earth surface :-

Note :-

Relation b/w Magnetic force and Coriolis force :-

$$\vec{F}_{cor} = 2m\,\vec{v}\times\vec{\omega}$$

$\omega = \omega(\cos\lambda\,\hat{j} \pm \cos\lambda\,\hat{k})$

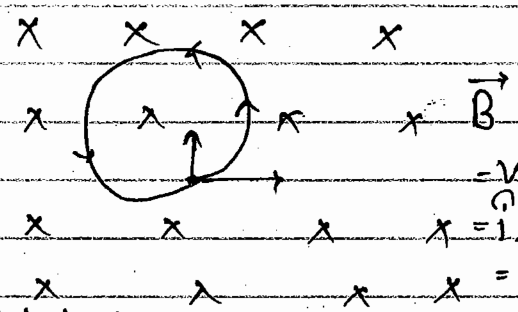


$\omega_\perp$ (due to +ve sign)

Clockwise

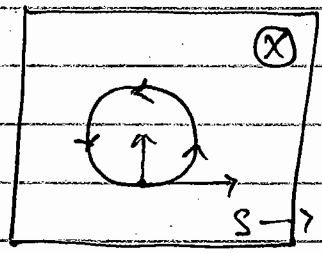
N — sphere

$\vec{F}_m = q\,\vec{V}\times\vec{B}$ {B = Constt}



$= \vec{V}\times\vec{B}$
$= \hat{i}\times(-\hat{k})$
$= \hat{j}$

Anticlockwise Path $\quad \vec{V}\perp\vec{B}$



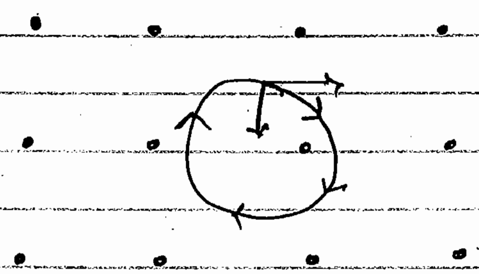
$\omega_\perp$ (due to negative sign)

Anti-Clockwise

S → sphere

If we ~~throw~~ thrown a charge perpendicular to the direction of field the charge move in a anti-clockwise circle.

*



Clock wise

If $\vec{V}\perp\vec{B}$ ⟶ Circle path

$\vec{V}$ is not perpendicular $\vec{B}$ ⟶ Helix

Conclusion :-

"Due to Coriolis force a horizontally thrown object will move clockwise in N-sphere and Anticlockwise in S-sphere."

* Radius of path followed by particle due to Coriolis force :-

In such cases :-

Centipital force = Coriolis force

$$\frac{\cancel{m} v^{\cancel{2}}}{r} = 2\cancel{m}\cancel{v}\omega_{\perp} \sin 90^{\circ}$$

$$r = \frac{v}{2\omega_{\perp}} = \frac{v}{2m \sin\lambda}$$

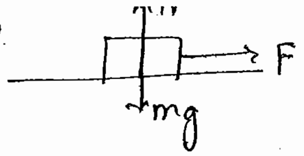
$$\boxed{r = \frac{v}{2\omega \sin\lambda}}$$

Deviation is minimun at equator for a vertically thrown particle.

* Coriolis and Centrifugal forces :-

Net 2011
Q.24

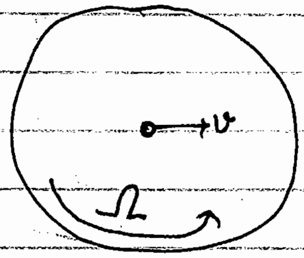
Normal force arises due to responce of perpendicular force



Ans

$\vec{F_{cf}} = m\omega^2 r_\perp$ ← Just after firing.

$r_\perp = 0$ { becoz initialy is at centre of rotating frame }



So $\boxed{\vec{F_{cf}} = 0}$

$\vec{F_{cor}} = 2m\vec{v} \times \vec{\omega}$

$|\vec{F_{cor}}| = 2mv\Omega \sin 90° = 2mv\Omega$

$a = \frac{Force}{Mass}$

$= \frac{2\cancel{m}v\Omega}{\cancel{m}}$

$= 2v\Omega$ to his right.

Ques In Q.N. 24 if the person is standing on the edge of the plateform and fires the bullet in radially inward direction what is the acc^n of prevous frame just after the firing, if radius of the plateform is R?

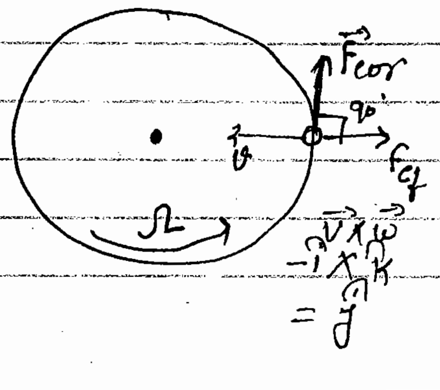
Sol^n

Angular velocity $= \Omega$
initial velocity $= v$

$r_\perp = R$

So

$F_{cf} = m\Omega^2 R$



Speed produced by rotation $v = \omega \times r$

$$\vec{F}_{cor} = 2m\,\vec{\omega} \times \vec{V}$$
$$= 2mv\Omega \sin 90^\circ = 2mv\Omega$$
$$\vec{F}_{cor} = 2mv\Omega$$

So Resultant force :-

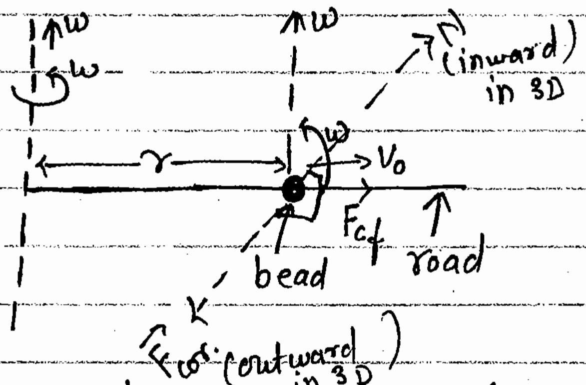
$$= \sqrt{\vec{F}_{cf}^{2} + \vec{F}_{cor}^{2}}$$
$$= m\sqrt{\Omega^4 R^2 + 4\Omega^2 V^2}$$

$$\vec{a} = \frac{\text{resultant force}}{\text{mass}}$$

$$\boxed{\vec{a} = \sqrt{\Omega^4 R^2 + 4\Omega^2 V^2}}$$

Ans

A-5
Q.16



$\vec{f}_{cor} = 2mV_0\omega$

∵ Here normal reaction arises due to coriolis forces.

Here F_{cor} is ⊥ to rod so normal reaction will arise in response to F_{cor}.

$$\therefore N = \vec{F}_{cor}$$

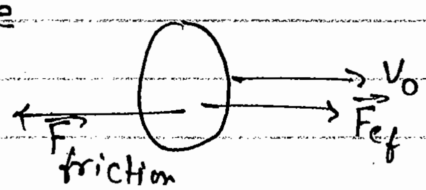
$$\boxed{N = 2mV_0\omega}$$

Ans

A-5
Q.17

Soln

Here speed v must be constant

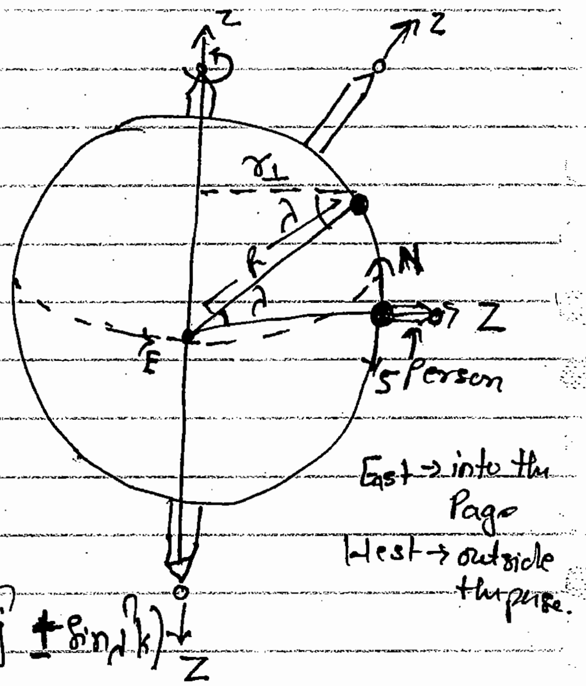


$\therefore$ $\vec{F}_{cf}$ is cancelled (balanced by) another force which is friction force.

$$\boxed{\vec{F}_r = \vec{F}_{cf} = m\omega^2 r}$$

A-5
Q.9

Soln



$r_\perp = R\cos\lambda$

$\lambda = 0$ (at equator)

$r_\perp = R$

$\vec{F}_{cf} = m\omega^2 R$

$\vec{F}_{cor} = 2m\vec{v}\times\vec{\omega}$

$= 2m(\pm v\hat{i}) \times \omega(\cos\lambda\,\hat{j} \pm \sin\lambda\,\hat{k})$

$= 2m(\pm v\hat{i}) \times \omega(\cos 90\,\hat{j} \pm \sin 90\,\hat{k})$

$\because$ $\lambda = 90$ at equator

$$\boxed{\vec{F}_{cor} = 2m\omega v\,(\pm\hat{k})}$$

quator is a latitude. Train moving
Along the latitude means train. move either East or West
for East +V for West -V.

If object moving towards east then $(+\hat{k})$ &
if moving toward west then $(-\hat{k})$.
the F_{cor} is radially outward and radially inward respectivily.

$$\vec{F}_{cf} = \vec{F}_{cor}$$

$$m\omega^2 R = 2m\omega V$$

$$\boxed{V = \frac{\omega R}{2}}$$

Ans

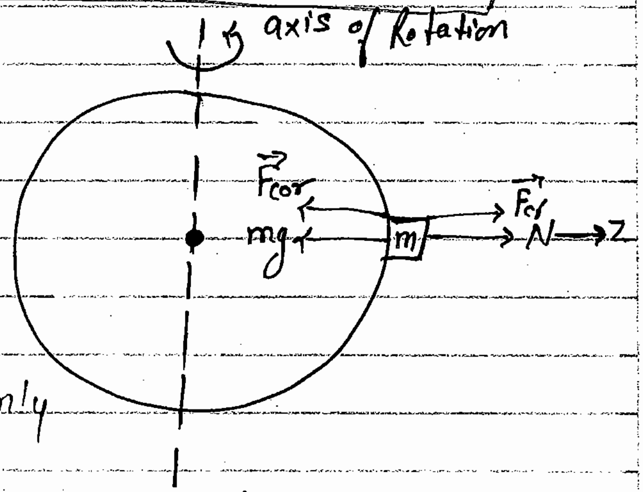
A-5
Q.10

Sol^n

$$\boxed{\text{Apparant Weight} = \text{Normal Reaction}}$$

OR

$$\boxed{\text{Reading of machine} = \text{Normal Reaction}}$$



In order to N = mg
then $\vec{F}_{cf} = \vec{F}_{cor}$
So mg → true weight
is equal to normal reaction.
{ In previous ques $\vec{F}_{cor} = \vec{F}_{cf}$ only when
Person is running from East to west.
so option (c) is correct.

Qus 12

Meridian Plane

Magnetic Maridian Plane

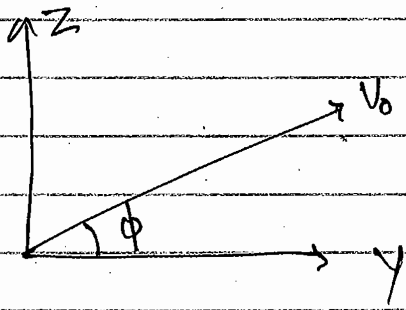
Geographic Meridian Plane.

Meridian Plane = A vertical plane.

When we take a horizontal plane in East - West direction then the meridian plane is a vertical plane $\perp^{or}$ to the horizontal plane in North - South direction.

$\vec{V} = V_0 (\cos\phi \hat{j} + \sin\phi \hat{k})$

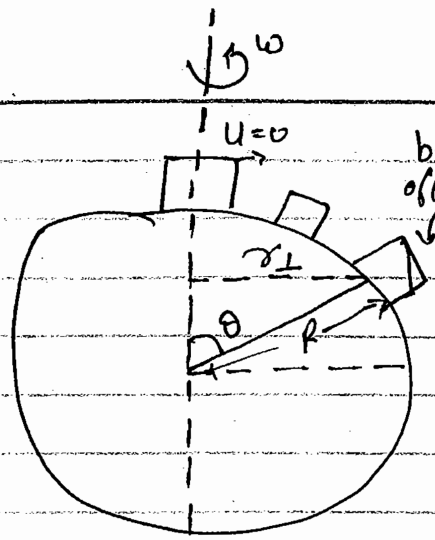
$\vec{\omega} = \omega (\cos\lambda \hat{j} + \sin\lambda \hat{k})$



(if not given hemisphere then always take Northarn hemisphere.

$\vec{F}_{cor} = 2m\vec{V} \times \vec{\omega}$

Q.14



$$\boxed{\theta = \cos^{-1}\frac{2}{3}}$$ → see the booklet.

Sol^n

$$\vec{F}_{cf} = m\omega^2 r_\perp$$

$$r_\perp = R\sin\theta$$

$$\vec{F}_{cf} = m\omega^2 R\sin\theta$$

$$= m\omega^2 R\sqrt{1-\cos^2\theta}$$

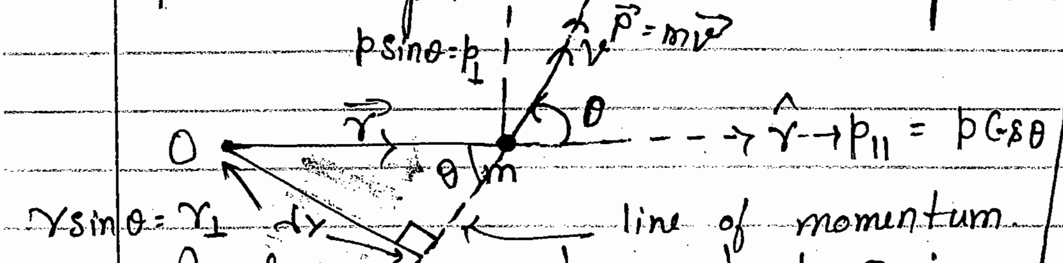
$$\vec{F}_{cf} = m\omega^2 R\sqrt{1-\frac{4}{9}}$$

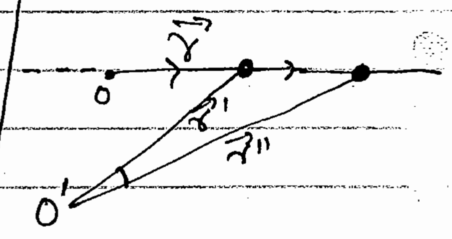
$$\boxed{\vec{F}_{cf} = \frac{\sqrt{5}}{3}\, m\omega^2 R}$$ Ans

{ it is a famous prob. and for θ see booklet }

* Angular Momentum And Torque :-

{ For a particle about a point } :-





Angular momentum about O :-

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$ ⇒ Vector form

$\vec{r}$ = position of particle from point O.

Use right hand thumb rule to find the direction of $\vec{L}$.

Magnitude form :-

$$\boxed{|\vec{L}| = rp\sin\theta = rp_\perp}$$

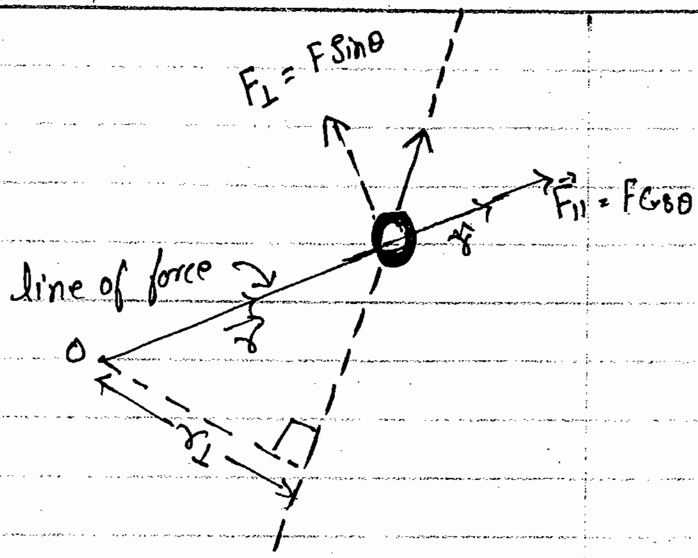
$(\because p_\perp = p\sin\theta)$

$$\therefore \boxed{|\vec{L}| = r_\perp p}$$

$(\because r_\perp = r\sin\theta)$

$$\therefore \boxed{L = rp\sin\theta = rp_\perp = pr_\perp}$$

* Torque :-



Qu A particle is moving in a circle of radius 'a' with constant speed v calculate its angular momentum about its centre of the circle.

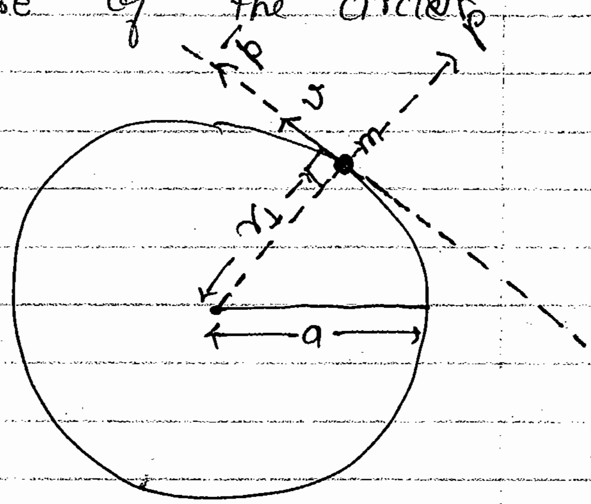
Solⁿ

$L = p\, r_\perp$

$$\boxed{L = mva}$$

$(\because p = mv) \quad r_\perp = a$

for Direction -



$\vec{L} = \vec{r} \times \vec{p}$ {finger curl from r to p}

~~So direction is up the plane of paper.~~ Up the plane of paper.

Qu A particle of mass m is moving with speed v_0 along a straight line $y = (\alpha x + \beta)$ $\alpha > 0$ $\beta > 0$ in x-y plane. Calculate angular momentum about origin.

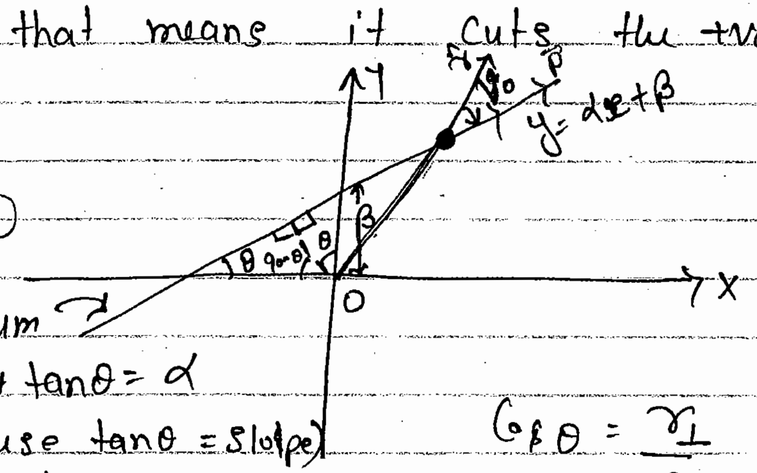
Sol^n

$$y = \alpha x + \beta$$

$\alpha \downarrow$ Slope, $\beta \downarrow$ intersept

Slope is +ve that means angle from +ve x direction is less than 90°.

Intersept is +ve that means it cuts the +ve Y-axis.



$$L = p r_\perp \quad \text{—} (1)$$

→ line of momentum

(Particle moving about this line)

$\tan\theta = \alpha$ (becouse $\tan\theta$ = slope) Which is α.

$$\cos\theta = \frac{r_\perp}{\beta}$$

$$\because \tan\theta = \frac{\alpha}{1} = \frac{\text{Perpendicular}}{\text{Base}} \quad \therefore r_\perp = \beta \cos\theta$$

$$\cos\theta = \frac{r_\perp}{\beta} = \frac{\text{Base}}{\text{Hypoten.}}$$

So $r_\perp = \beta \cdot \dfrac{\text{Base}}{\text{Hypoten.}} = \beta \cdot \dfrac{1}{\sqrt{\alpha^2+1}}$

So $\boxed{r_\perp = \dfrac{\beta}{\sqrt{\alpha^2+1}}}$

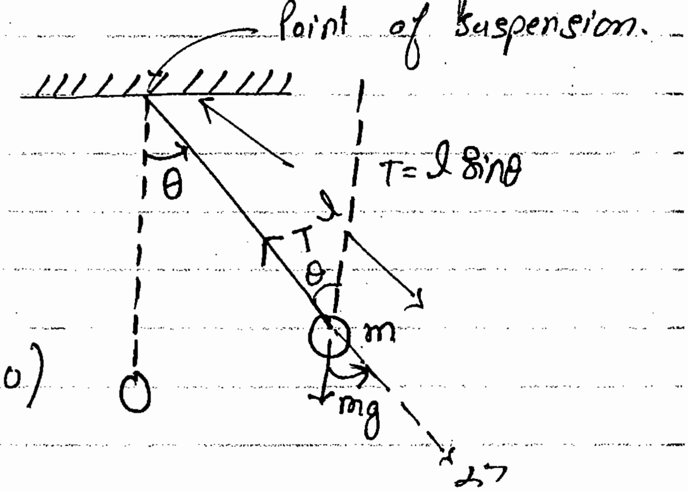
So $\boxed{L = m v_0 \dfrac{\beta}{\sqrt{\alpha^2+1}}}$ {from (1)}

direction of angular momentum is into the page or (−z) axis.

$r \times p$

Que A simple pendulum consisting of a small bob of mass m and light string of length l is deflected by an angle θ from lowest position. Calculate torque about point of suspention in this position.

Sol^n



$\tau = F\, r_\perp$

① Due to tension :-

$\tau = T.\,0 = 0 \quad (\because r_\perp = 0)$

$\boxed{\tau = 0}$

If a force passes through a point so that force will not produce any torque about point.

② Due to mg :-

$\tau = mg.\, l\sin\theta$

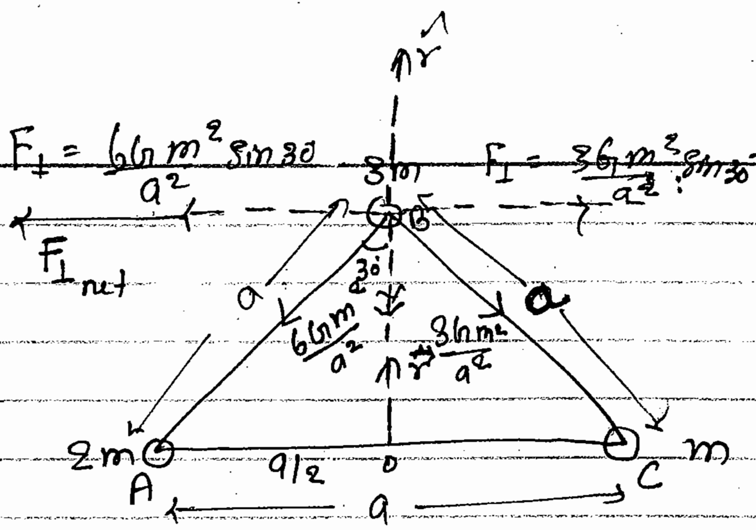
$\boxed{\tau = mgl\sin\theta} \quad (r_\perp = l\sin\theta)$

Direction of τ is into the plane of paper.

Que. What is torque on mass 3m due to forces of other masses about point O ?

Sol^n

(1) $\tau = F_\perp r$



$r = \frac{\sqrt{3}}{2} a$

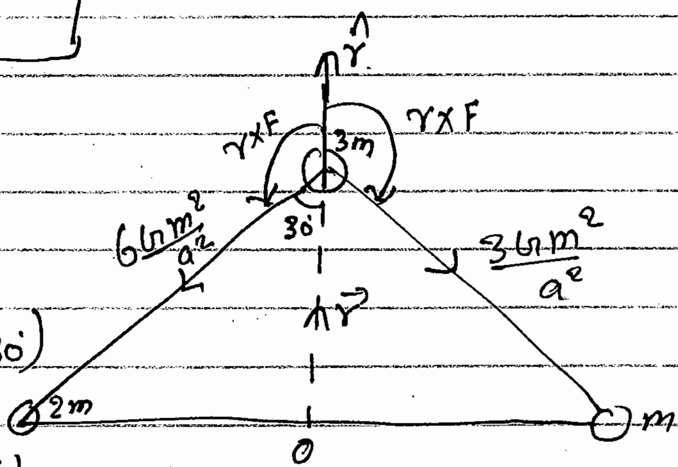
$F_{\perp net} = \frac{6m^2G}{a^2}\sin 30° - \frac{3Gm^2}{a^2}\sin 30°$

$F_{\perp net} = \frac{3Gm^2}{a^2}\sin 30°$

So $\tau = \frac{3Gm^2}{a^{\cancel{2}}} \cdot \frac{\sqrt{3}}{2}\cancel{a} \cdot \frac{1}{2}$

$$\boxed{\tau = \frac{3\sqrt{3}Gm^2}{4a}}$$

Second Method :-



$\tau = rF\sin\theta$

$= \frac{\sqrt{3}}{2}a \frac{6Gm^2}{a^2}\sin(\pi - 30°) - \frac{\sqrt{3}}{2}a\frac{3Gm^2}{a^2}\sin(\pi - 30°)$

$\tau = \frac{\sqrt{3}}{2}a\frac{3Gm^2}{a^2}\{2-1\}\sin(\pi-30°)$

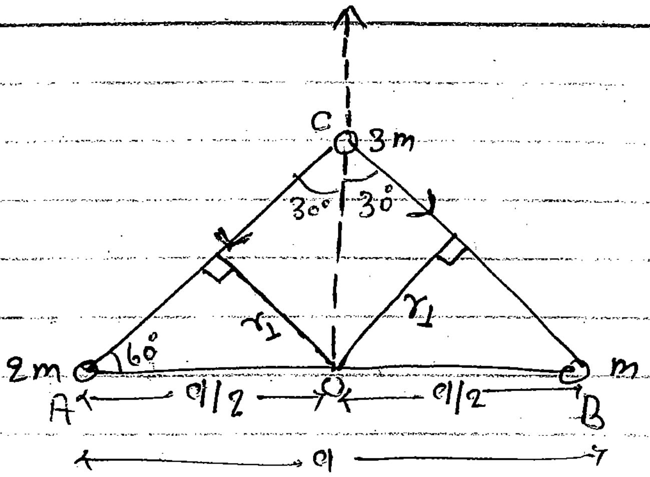
$= \frac{3\sqrt{3}}{2}\frac{Gm^2}{a}\sin 30° = \frac{3\sqrt{3}}{2}\frac{Gm^2}{a}\cdot\frac{1}{2}$

$$\boxed{\tau = \frac{3\sqrt{3}}{4}\frac{Gm^2}{a}}$$ up the plane of paper.

A

Third Method :-

$$\tau = \vec{F} r_{\perp}$$



$$\sin 60° = \frac{r_{\perp}}{a/2}$$

$$\frac{\sqrt{3}}{2} = \frac{2r_{\perp}}{a}$$

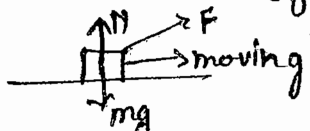
$$\boxed{r_{\perp} = \frac{\sqrt{3}a}{4}} \quad , \quad \vec{F} =$$

$$\vec{F} = \frac{6Gm^2}{a^2} \sin 30°$$

$$\tau = \vec{F} r_{\perp}$$

$$\tau =$$

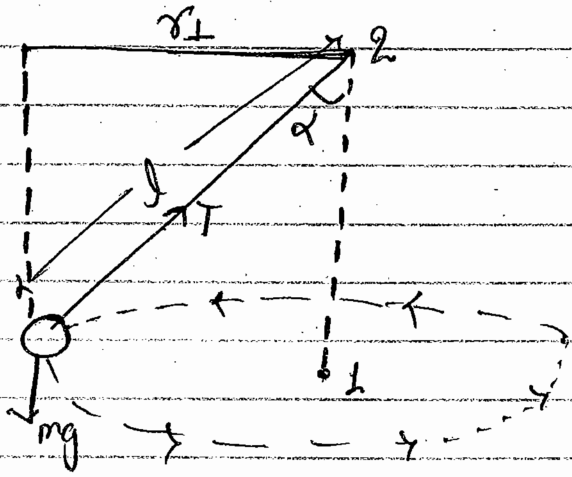
force is resolve in the direction of motion.



28/Aug/2014

Q. A conical pendulum consist of a bob of mass m and light string of length l. If string of pendulum makes an angle α with downward vertical.

Solⁿ Torque about point 2:-



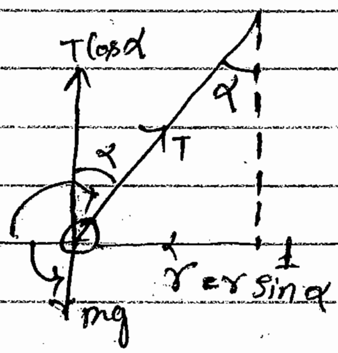
Here tension is not produces Torque because it passes through the point of suspension.

So $\tau_2 = mg\, r_\perp$

$$\boxed{\tau_2 = mg\,l\sin\alpha}$$

About point 1:-

$\tau = \vec{F}_\perp r$, $\tau = F, r_\perp$



$\tau_1 = mgl\sin\alpha - T\cos\alpha . l\sin\alpha$

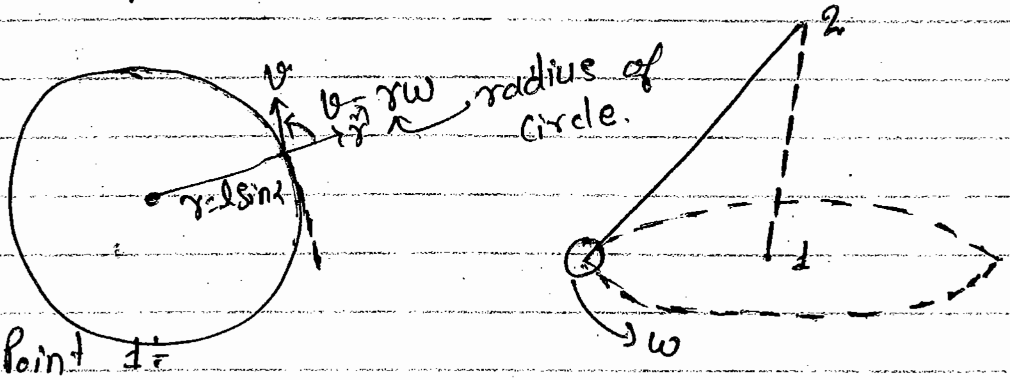
$$\boxed{\tau_1 = l\sin\alpha\,(mg - T\cos\alpha)}$$

$\because$ Here there is vertical equilibrium.

So $\because T\cos\alpha = mg$.

$\vec{L} = \vec{r} \times \vec{p}$

Ques In previous question if angular speed of conical pendium is ω Calculate angular momentum about points 1 and 2.



About Point 1 :-

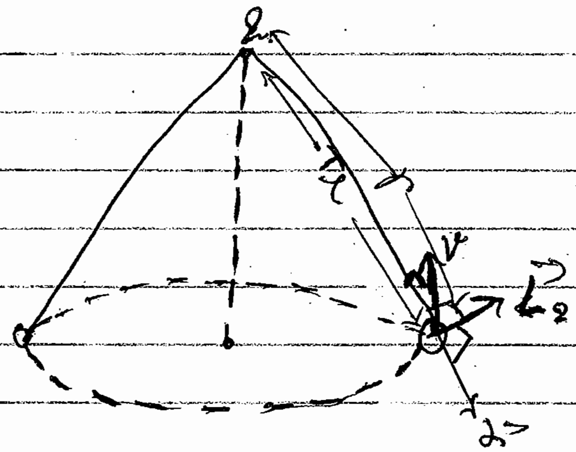
$L = p \cdot r_\perp$

$= mv \cdot r$

$= m\omega r^2$

$\boxed{L_1 = m\omega l^2 \sin^2\alpha}$ (And direction of angular momentum along the vertical line.)

About point 2 :-



$L_2 = p r_\perp$

$= mv r_\perp$

$= m\omega l \sin\alpha \cdot r_\perp$

$= m\omega l \sin\alpha \cdot l$

$\boxed{L_2 = m\omega l^2 \sin\alpha}$

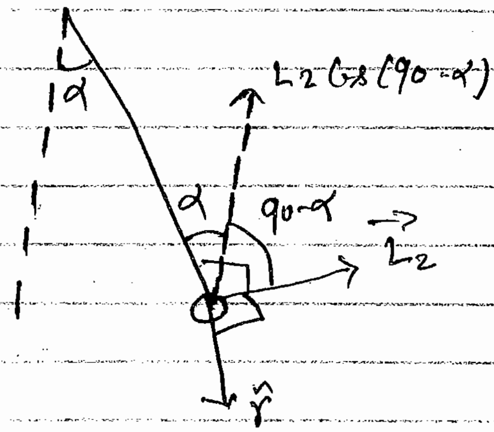
It is not along the vertical line. It is perpendicular to string.

Let us take component of L_2 along vertical line,

L_2 (along vertical line) =

$= L_2 \sin\alpha$

$= m\omega l^2 \sin\alpha . \sin\alpha$



$$\boxed{L_2 = m\omega l^2 \sin^2\alpha = L_1}$$

Conclusion :-

If we calculate angular momentum about different points of axis of rotation then magnitude as well as direction will be different. However value of angular momentum about or along the axis [i.e. Component along the axis is always same].

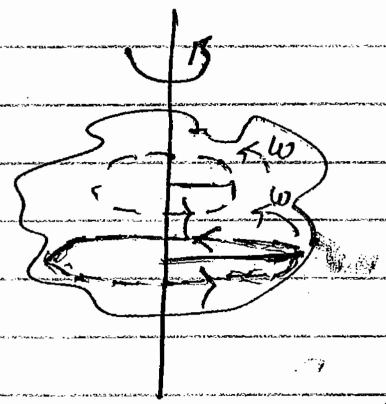
And its value is given as-

$$\boxed{L_{axis} = m\omega r_\perp^2} \quad \left\{ \text{for Single Particle} \right\}$$

Where $r_\perp$ = perpendicular distance from axis of Rotation

* If there is a rigid body :-

So every particle moves in a circle of different radius, but same angular speed.



For a rigid body :-

$$L_{axis} = \sum_i m_i \omega r_{\perp i}^2$$

$$\boxed{L_{axis} = \omega \sum_{i=1}^{n} m_i r_{\perp i}^2}$$ for system of particles.

This may or may not be total angular momentum.

$$\boxed{L_{axis} = \omega I}$$

$$I = \sum_{i=1}^{N} w_i r_{\perp i}^2 = \text{Moment of Inertia}$$

* Moment of Inertia about an axis:-

$$\boxed{I = \sum_{i=1}^{N} m r_{\perp i}^2}$$ → for descreate case

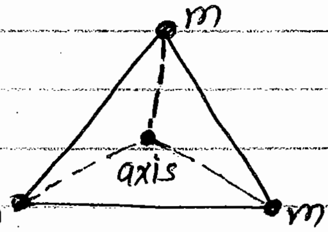
$$\boxed{I = \int dm\, r_{\perp}^2}$$ → Continuous case

$r_\perp$ = $\perp^r$ distance of elementary part dm from axis.

Q. Three particles of mass m are placed at the corners of an equilateral triangle of side 'a'. Calculate M.I. about axis passes through centroide and $\perp^{r}$ to its plane.

Sol^n

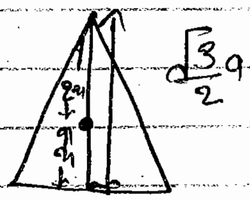
$$I = \sum_{i=1}^{3} m_i r_{\perp i}^2$$



$$I = m\frac{a^2}{3} + m.\frac{a^2}{3} + m.\frac{a^2}{3}$$

$$= \frac{\cancel{3} ma^2}{\cancel{3}}$$

$$\boxed{I = ma^2}$$



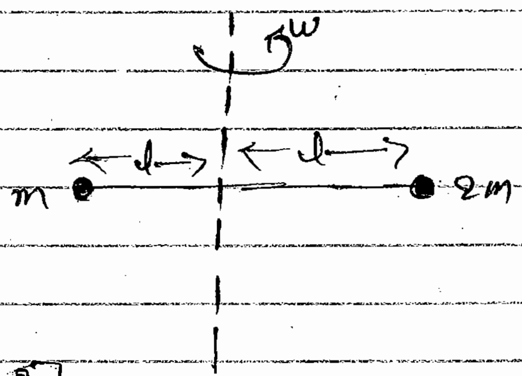
$$3x = \frac{\sqrt{3}}{2}a$$

$$x = \frac{a}{2\sqrt{3}}$$

$$2x = \frac{a}{\sqrt{3}}$$

Jet-2011

Q. What is angular momentum about axis of rotation?



$$L_{axis} = I\omega$$

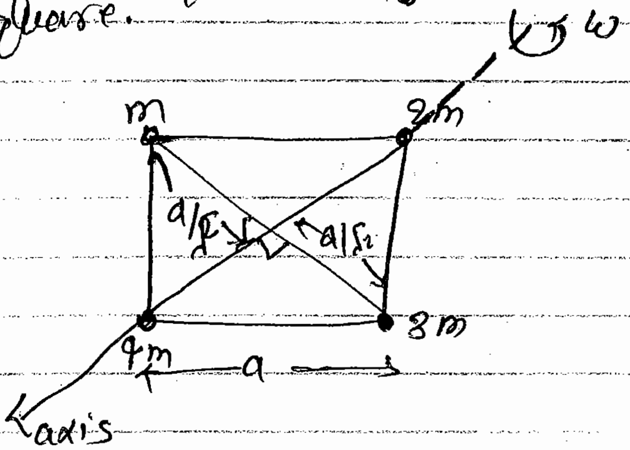
$$= \omega\left[\sum_{i=1}^{2} m_i r_{\perp i}^2\right]$$

$$= \omega\left[m \times l^2 + 2m l^2\right]$$

$$= 3ml^2\omega$$

Q. Four particles are placed at corners of square as shown in fig.
Calculate M.I. of system about diagonal of square.

Sol^n



$$I = \sum_{i=1}^{4} m_i r_{\perp i}^2$$

$$= m \times \frac{a^2}{2} + 2m \times 0 + 3m \times \frac{a^2}{2} + 4m \times 0$$

$$= 4m \times \frac{a^2}{2} = 2ma^2$$

$$\boxed{I = 2ma^2}$$

Note :-

Mass density :-

Linear $(\lambda) = \frac{\text{mass}}{\text{length}} = \frac{m}{l} = \frac{dm}{dl}$ → non-Uniform.

Surface $(\sigma) = \frac{\text{mass}}{\text{area}} = \frac{dm}{dA} = \frac{m}{A}$ → Uniform

Volume $(\rho) = \frac{\text{mass}}{\text{Volume}} = \frac{dm}{dV} = \frac{m}{V}$

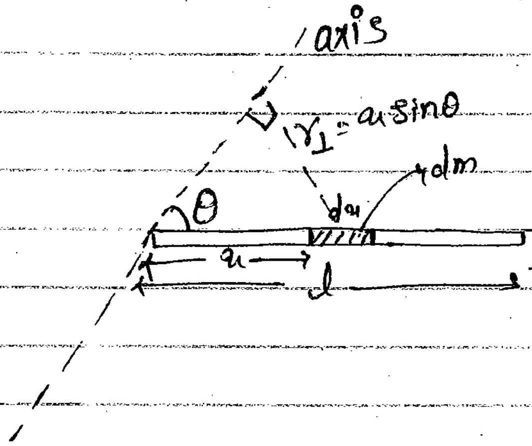
* M.I. of a Continuous object :-

A thin rod {Uniform} :-

What is moment of Inertia about axis shown in figure.

Mass of rod = m

length of rod = L



$$I = \int dm\, r_\perp^2$$

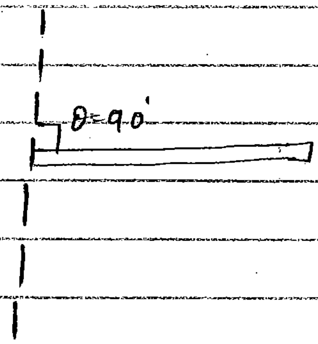
$$= \int_0^L \frac{M}{L} dx \cdot x^2 \sin^2\theta$$

$$= \frac{M}{L}\sin^2\theta \int_0^L x^2\, dx = \frac{ML^2}{3}\sin^2\theta$$

Case I :-

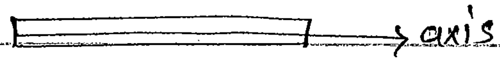
If $\theta = 90°$

then $\boxed{I = \frac{ML^2}{3}}$



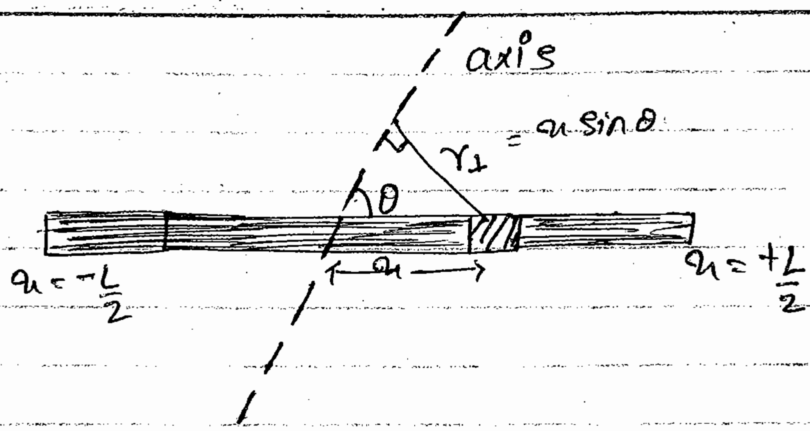
Case II :-

If $\theta = 0°$



then $\boxed{I = 0}$

Case III :-



$$I = \int_{-L/2}^{+L/2} dm\, r_\perp^2$$

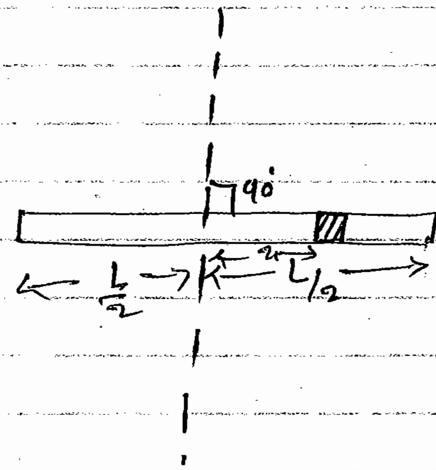
$$= \int_{-L/2}^{L/2} \frac{M}{L}\, dx \cdot x^2 \sin^2\theta$$

$$= \frac{M}{L}\sin^2\theta \int_{-L/2}^{+L/2} x^2\, dx = \frac{M}{L}\sin^2\theta \left[\frac{x^3}{3}\right]_{-L/2}^{+L/2}$$

$$= \frac{1}{3}\frac{M}{L}\sin^2\theta\left[\frac{L^3}{8} + \frac{L^3}{8}\right] = \frac{1}{3}\frac{M}{\cancel{L}}\sin^2\theta\left[\frac{2L^{\cancel{3}}}{\cancel{8}\,4}\right]$$

$$\boxed{I = \frac{ML^2\sin^2\theta}{12}}$$

Case - IV :-



Q. Four thin rods adjoit to get to form a square loop. If mass of thin rod is M and length is L. What is moment of inertia of square plane about it ~~square~~ diagonal plane.

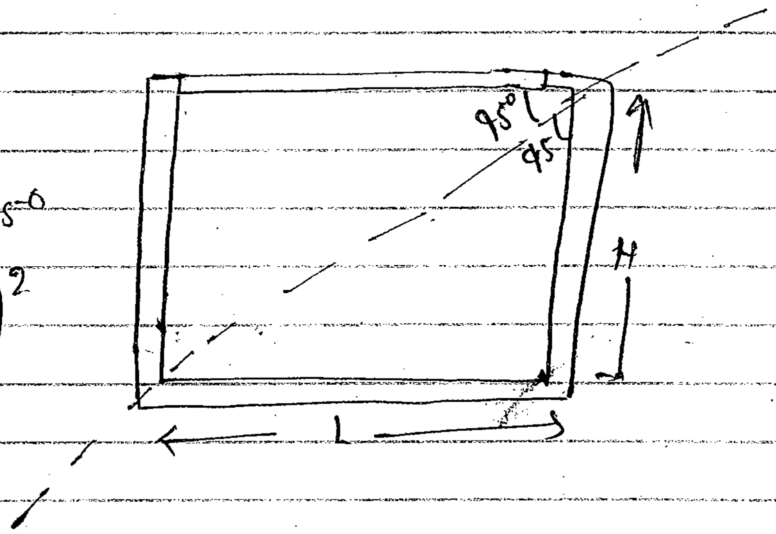
Sol

$$I = \frac{ML^2}{3}\sin^2\theta$$

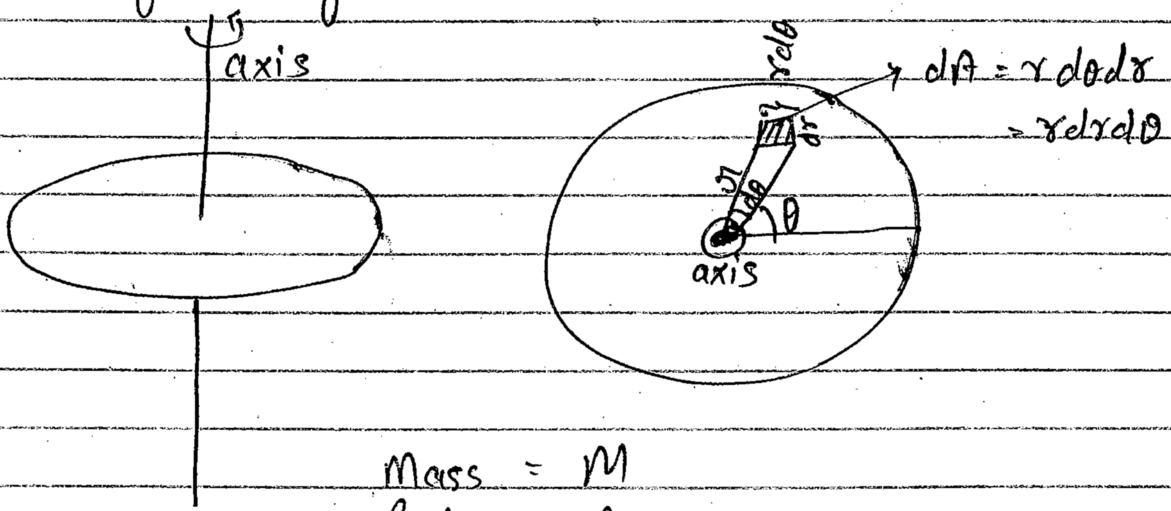
$$= 4 \times \frac{ML^2}{3}\sin^2 45^\circ$$

$$= 4 \times \frac{ML^2}{3} \times \left(\frac{1}{\sqrt{2}}\right)^2$$

$$\boxed{I = \frac{2ML^2}{3}}$$



Q. M.I. of Uniform disc :-



Mass = M
Radius = R

$$\sigma = \frac{M}{\pi R^2}$$

$$\sigma = \frac{dm}{dA}$$

$$\frac{M}{R^2} = \frac{dM}{dA}$$

$$\boxed{dm = \frac{M}{R^2} \cdot dA}$$

$$I = \int dm\, r_{\perp}^2$$

$$= \int \frac{M}{\pi R^2} dA \cdot r^2$$

$$= \frac{M}{\pi R^2} \iint r\, dr\, d\theta \cdot r^2$$

$$= \frac{M}{\pi R^2} \int_0^R r^3 dr \int_0^{2\pi} d\theta$$

$$= \frac{\cancel{2\pi} M}{\cancel{\pi} \cancel{R^2}} \cdot \frac{R^{\cancel{4}}}{\cancel{4}}$$

$$\boxed{I = \frac{MR^2}{2}}$$

{ Memorize Read list of booklet. }

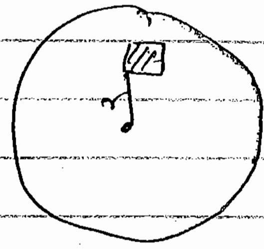
Que A circular disc of mass M and Radius R have non-uniform density varying a distance from center as $\sigma = \sigma_0\left(1 - \frac{r}{R}\right)$ Calculate moment of inertia of disc in terms of A and R about $\perp^{or}$ axis through it centre.

Sol^n

$$\because \sigma = \frac{dM}{dA}$$

$$\sigma_0\left(1 - \frac{r}{R}\right) = \frac{dM}{dA}$$

$$dM = \sigma_0\left(1-\frac{r}{R}\right)dA$$



$$I = \int dM\, r_\perp^2$$

$$= \int \sigma_0\left(1-\frac{r}{R}\right)\cdot r^2 \cdot dA$$

$$= \iint \sigma_0\left(1-\frac{r}{R}\right) r\,dr\,d\theta \cdot r^2$$

$$= \sigma_0 \int_0^R \left(1-\frac{r}{R}\right) r^3\,dr \int_0^{2\pi} d\theta$$

$$= 2\pi\sigma_0\left[\int_0^R r^3\,dr - \int_0^R \frac{r^4}{R}\,dr\right]$$

$$= 2\pi\sigma_0\left[\frac{R^4}{4} - \frac{R^5}{5R}\right]$$

$$= 2\pi\sigma_0\left[\frac{R^4}{4} - \frac{R^4}{5}\right]$$

$$= \cancel{2}\pi\sigma_0\left[\frac{R^4}{\cancel{20}}\right]$$

$$\boxed{I = \frac{\pi\sigma_0 R^4}{10}} \quad \text{—— } (A)$$

To eliminate σ_0 integrate dM :-

$$\int dM = \sigma_0 \int_0^R \left(1-\frac{r}{R}\right) r\,dr \int_0^{2\pi} d\theta$$

$$= 2\pi\sigma_0\left[\frac{R^2}{2} - \frac{R^3}{3R}\right]$$

$$= \sigma_0 2\pi \cdot \frac{R^2}{6}$$

$$M = \frac{\sigma_0 \pi R^2}{3}$$

$$\boxed{\sigma_0 = \frac{3M}{\pi R^2}}$$

Put in (A)

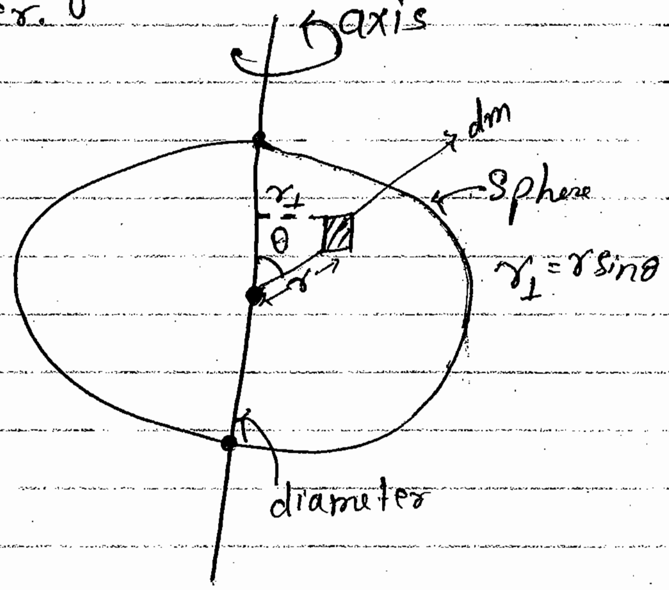
$$I = \frac{3M}{\cancel{\pi} R^{\cancel{2}}} \cdot \frac{\cancel{\pi} R^{\cancel{4}}}{10}$$

$$\boxed{I = \frac{3MR^2}{10}}$$

Ans

Q. A solid sphere of mass M and radius R has volume mass density $\rho = Kr^2$ where K is constant. r is distance from centre calculate M.I. about its diameter.

Sol^n



$$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\rho = \frac{dm}{dv}$$

$$Kr^2 = \frac{dm}{dv}$$

$$dm = Kr^2 \cdot r^2 dr \cdot \sin\theta d\theta \cdot d\phi$$

$$= Kr^4 dr \sin\theta d\theta \cdot d\phi$$

$$I = \int dm \, r_\perp^2$$

$$= \int dm \, r^2 \sin^2\theta$$

$$I = K\int_0^R r^6\,dr\int_0^\pi \sin^3\theta\,d\theta\int_0^{2\pi} d\phi$$

$$= \frac{KR^7}{7}\cdot 2\pi\cdot\int_0^\pi \sin^2\theta\cdot\sin\theta\,d\theta$$

$$\int_0^\pi \sin^3\theta$$

$$\int_0^\pi (1-\cos^2\theta)\sin\theta\,d\theta$$

$$-\int_{+1}^{-1}(1-t^2)\,dt$$

$$= \frac{KR^7}{7}\cdot 2\pi\int_{-1}^{+1}(1-t^2)\,dt$$

$$= \frac{KR^7}{7}\cdot 2\pi\left[\left[t\right]_{-1}^{+1} - \left[\frac{t^3}{3}\right]_{-1}^{+1}\right]$$

$$= \frac{KR^7}{7}\cdot 2\pi\left[1+1\right] - \left[\frac{1}{3}+\frac{1}{3}\right]$$

$$= \frac{KR^7}{7}\,2\pi\cdot\left(2-\frac{2}{3}\right) = \frac{KR^7}{7}\,2\pi\cdot\left(\frac{6-2}{3}\right)$$

$$= \frac{8\pi KR^7}{7\times 3} = \frac{8\pi KR^7}{21}$$ Ans — (a)

Now to remove K

$$\int dm = K\int_0^R r^4\,dr\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi} d\phi$$

$$m = \frac{KR^5}{5}\cdot\left[-\cos\theta\right]_0^\pi\cdot 2\pi$$

$$= \frac{2\pi KR^5}{5}\cdot 2$$

$$m = \frac{4\pi KR^5}{5}$$

$$\boxed{K = \frac{5M}{4\pi R^5}}$$

Put the value of k in eqⁿ (a)

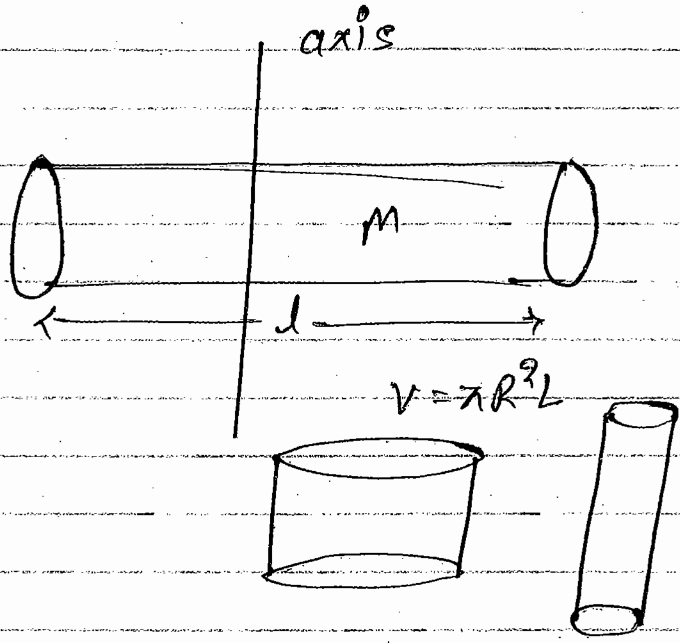
$$I = \frac{\cancel{8}^{2}\pi R^7}{21} \times \frac{5M}{\cancel{4}\pi R^5}$$

$$= \frac{10MR^{\cancel{7}}}{21R^{\cancel{5}}} = \frac{10MR^2}{21}$$

$$\boxed{I = \frac{10MR^2}{21}}$$

Ans

Q.4 What should be ratio of radius and length of a solid cylinder of uniform mass density so that its moment of inertia through the axis and $\perp^{or}$ to its length is minimum for given volume.



Solⁿ

$$I = \frac{MR^2}{4} + \frac{ML^2}{12}$$

given V = Constt.
M = Constt.

$$V = \pi R^2 l$$

$$R^2 = \frac{V}{\pi l}$$

$$I = M\left[\frac{R^2}{4} + \frac{l^2}{12}\right]$$

$$I = M\left[\frac{V}{4\pi l} + \frac{l^2}{12}\right]$$

$$I = f(l)$$

for I to be minimum -

$$\frac{dI}{dl} = 0$$

$$0 = M\left[\frac{-V}{4\pi l^2} + \frac{l}{6}\right]$$

$$\frac{l}{6} = \frac{V}{4\pi l^2}$$

$$\frac{\pi R^2 l}{4\pi l^2} = \frac{l}{6}$$

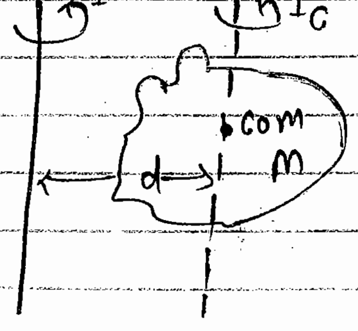
$$\left(\frac{R}{L}\right)^2 = \frac{4}{6}$$

$$\boxed{\frac{R}{L} = \frac{2}{\sqrt{6}}}$$

* Parallel Axis Theorem :-

Applicable for 1d, 2d and 3d object. We take two parallel axes one of them must pass through centre of mass.

$$\boxed{I = I_c + Md^2}$$



M = mass of object

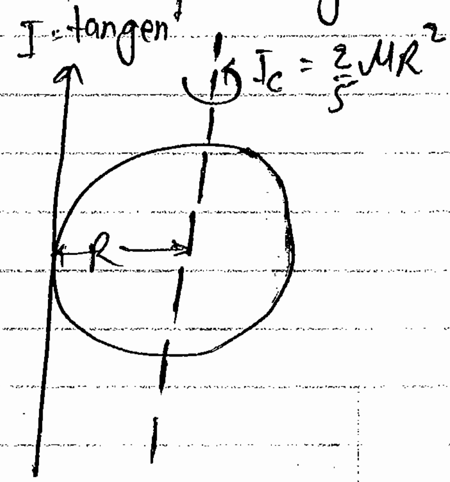
d = $\perp^{r}$ distance b/w the axis.

Application :-

1 M.I of a uniform sphere about its tangent

M = Mass of sphere

R = Radius of sphere

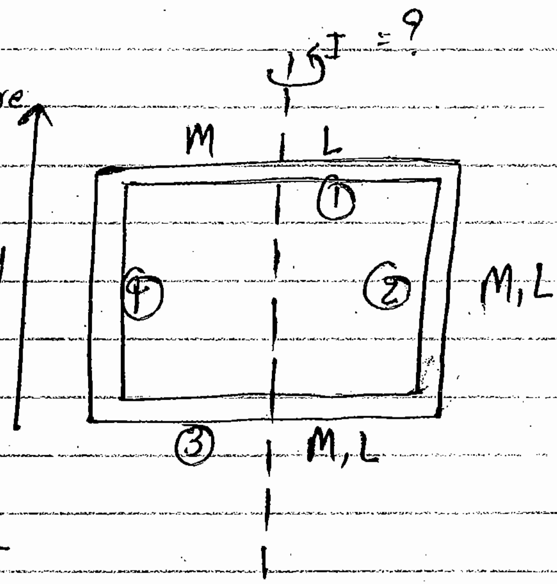


$$I_{tangent} = I_C + Md^2$$

$$= \frac{2}{5} MR^2 + MR^2$$

$$\boxed{I_{tangent} = \frac{7}{5} MR^2}$$

2 Calculate M.I. of square frame shown in fig. about an axis lying its plane and passing through the centre.



Sol^n M.I. about (1) Rod :-

$$I_{①} = \frac{ML^2}{12} \quad \text{--- (I)}$$

M. I. about (3) Rod :-

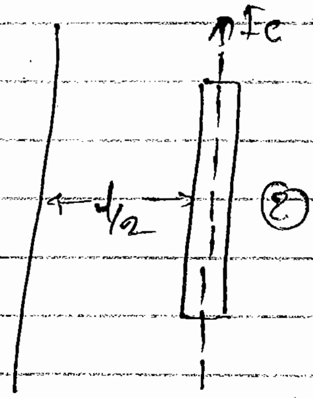
$$I_3 = \frac{ML^2}{12} \quad \text{--- (II)}$$

M.I. about (2) Rod

$I_{2} = I_c + Md$

$= 0 + \frac{ML^2}{4}$

$$\boxed{I_{2} = \frac{ML^2}{4}} \quad \text{—— (III)}$$



Similarly

$$I_4 = \frac{ML^2}{4} \quad \text{—— (IV)}$$

So $I_{Net} = I_1 + I_2 + I_3 + I_4$

$= \frac{ML^2}{12} + \frac{ML^2}{12} + \frac{ML^2}{4} + \frac{ML^2}{4}$

$= \frac{2ML^2}{12} + \frac{2ML^2}{4} = ML^2\left[\frac{2}{\cancel{12}_6} + \frac{2}{\cancel{4}_2}\right]$

$I_{net} = ML^2\left[\frac{2+6}{12}\right] = ML^2\left[\frac{\cancel{8}\,4\,2}{\cancel{12}\,\cancel{6}\,3}\right]$

$$\boxed{I_{net} = \frac{2}{3}ML^2}$$ Ans

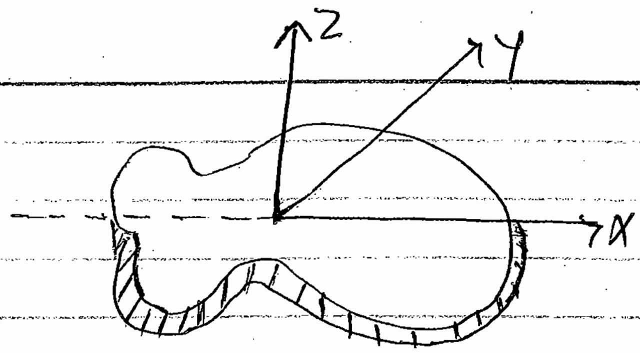
* Perpendicular Axis Theorem :-

Here we have three perpendicular axis (x, y, z). If x and y lie in the plane of object and z is perpendicular to its plane z must pass through point of intersection of x and y

⇒ It is not applicable in 3 dimension.

01/Sep./2014

$$\boxed{I_z = I_x + I_y}$$

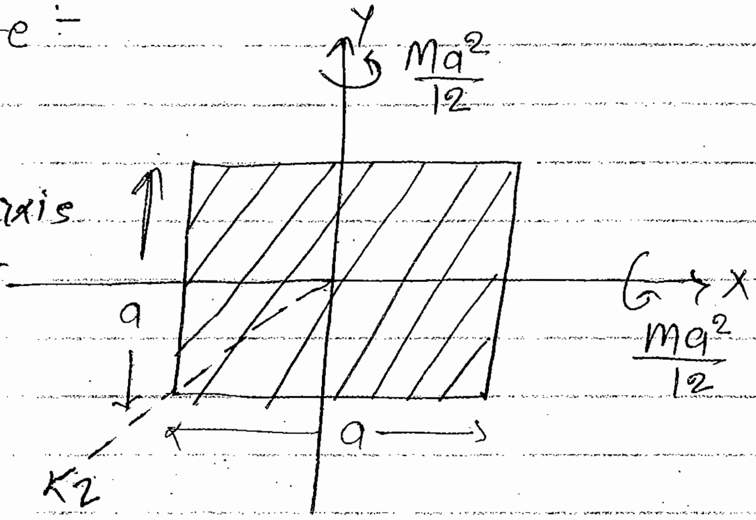


* Square Plate :-
{2-D object} :-

M.I about ⊥r axis through center :-

$$I_z = I_x + I_y$$

$$\boxed{I_z = \frac{Ma^2}{6}}$$



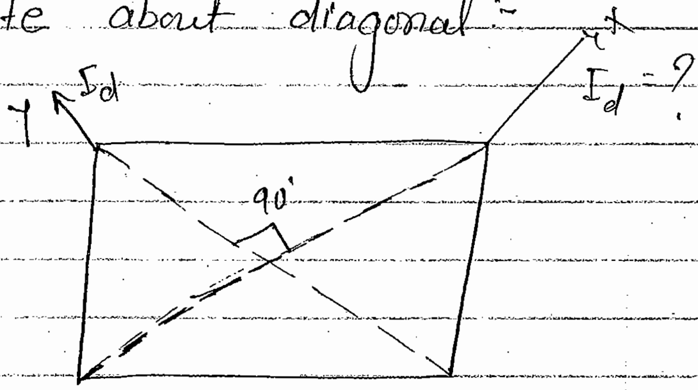
* M.I of square plate about diagonal :-

$$I_z = I_x + I_y$$

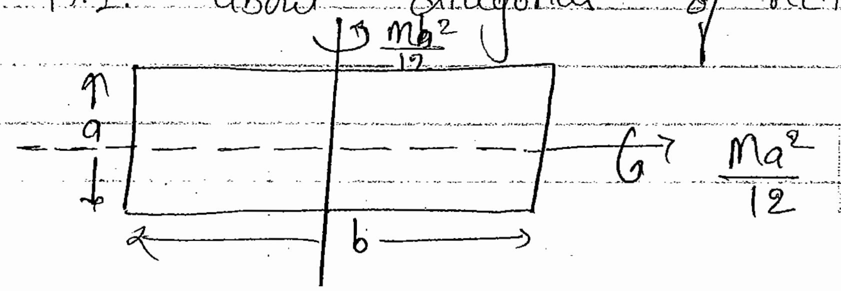
$$\frac{Ma^2}{6} = I_d + I_d$$

$$2 I_d = \frac{Ma^2}{6}$$

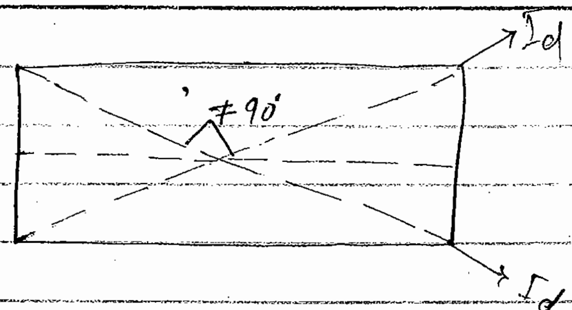
$$\boxed{I_d = \frac{Ma^2}{12}}$$



* What is the M.I. about diagonal of rectangle :-



I_d can not be calculated using $\perp^{or}$ axis theorem about its center.

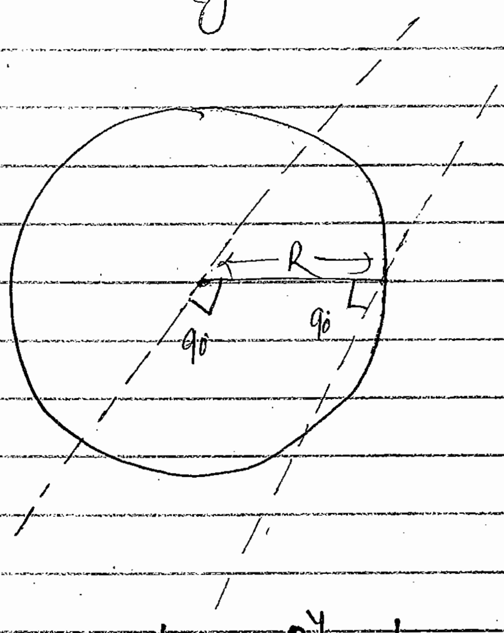


Note :- Inertia Tensor will be used.

$*$ Disc of mass M and Radius R :-
{Assume Disc is uniform}

Q. M.I about an axis $\perp^{or}$ to its surface and passing through its edge?

Sol^n Method I :-



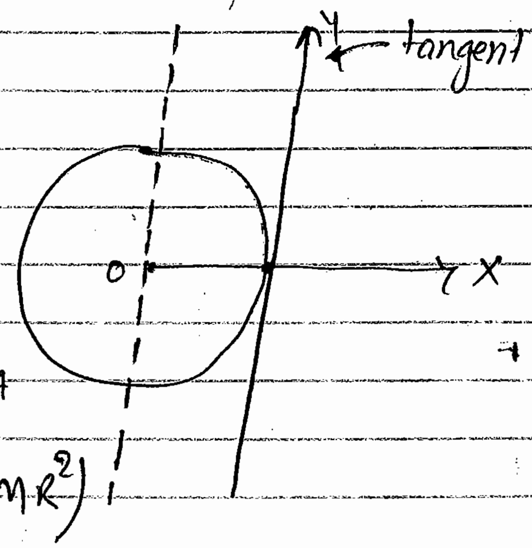
From $\parallel^{lel}$ Axis theorem :-

$$I = I_c + Md^2$$

$$= \frac{MR^2}{2} + MR^2$$

$$= \frac{3}{2} MR^2$$

Method II^{nd} :-



$$I_z = I_x + I_y$$

$$I_{required} = I_{diameter} + I_{tangent}$$

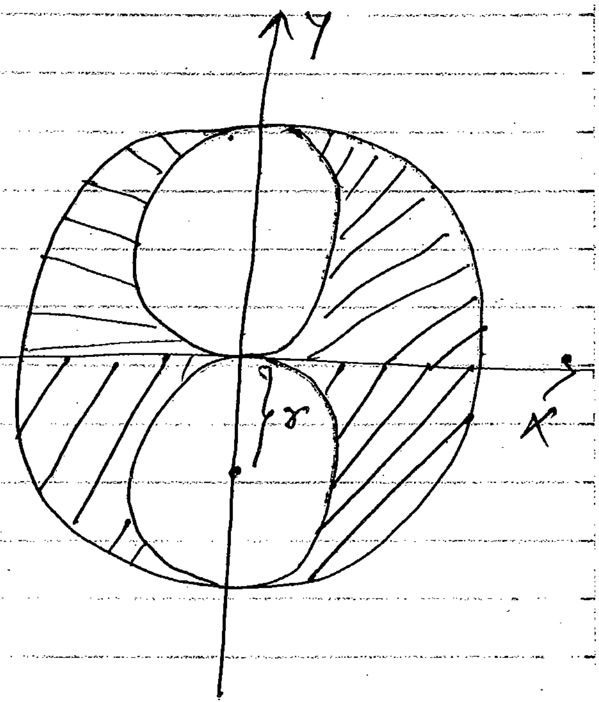
$$= \frac{MR^2}{4} + \left(\frac{MR^2}{4} + MR^2\right)$$

M.I. is a additive quantity.

$$= \frac{MR^2}{2} + MR^2 = \frac{3}{2} MR^2$$

So $\boxed{I_{required} = \frac{3}{2} MR^2}$ Ans

Date
Q.15



M.I. of remaining object = M.I. of big sphere − M.I. of two small sphere.

$$I_y = \frac{2}{5} MR^2 - \frac{2}{5} mr^2 \times 2$$

$$M = \rho \frac{4}{3} \pi R^3$$

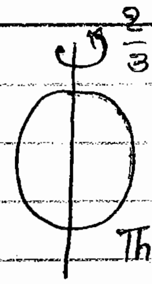
$$= \rho \frac{4}{3} \pi r^3, \quad r = \frac{R}{2}$$

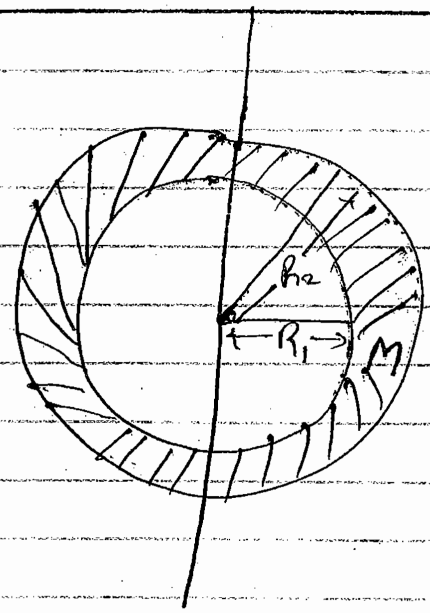
$$I_y =$$

$$I_x = \frac{2}{5} MR^2 - \frac{7}{5} mr^2 \times 2$$

Q.9 A thick hollow sphere of mass 'M' has inner and outer radii R_1 and R_2. Moment of inertia of sphere about its diameter is −?

$\underline{Sol^n}$





To check options –

(c) $\frac{1}{3} M(R_1^2 + R_2^2)$

for the thin sphere –

$R_1 = R_2 = R$

$$\boxed{I = \frac{2}{3} MR^2}$$

But for solid sphere –

$R_1 = 0 \quad R_2 = R$

$$\boxed{I = \frac{1}{3} MR^2}$$

which is not equal to the M.I. of solid sphere.

(d) $\frac{2}{5} M\left(\frac{R_2^5 - R_1^5}{R_2^3 - R_1^3}\right)$ $\left[\frac{0}{0} = \text{L'Hospital's rule.} \right.$ (होपिटलर))

$R_1 = R$

$R_2 = R + a \qquad \lim_{a \to 0}$

$$I = \frac{2}{5} M\left(\frac{R_2^5 - R_1^5}{R_2^3 - R_1^3}\right)$$

$$= \lim_{a \to 0} M \frac{2}{\cancel{5}} \frac{\cancel{5}}{3}\left(\frac{(R+a)^4 - 0}{(R+a)^2 - 0}\right)$$

$$= \frac{2}{3} M \frac{R^4}{R^2} = \frac{2}{3} MR^2$$

$$\boxed{I = \frac{2}{3} MR^2}$$

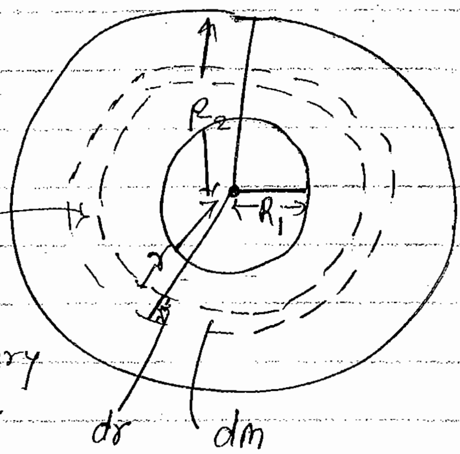
So it is correct.

* Moment of Inertia of a big object is to be calculated, we can first calculate M.I. of a small elementary object and integrate it.

Ex-

$$I = \int dI$$

$$dI = \frac{2}{3} dm r^2$$



$$\frac{dm}{dV} = \rho = \frac{M}{\frac{4}{3}\pi(R_2^3 - R_1^3)}$$

$$dm = \frac{3M \cdot dV}{4\pi(R_2^3 - R_1^3)}$$

$$\boxed{dm = \frac{3M}{4\pi(R_2^3 - R_1^3)} dV}$$

$$dV = 4\pi r^2 dr$$

$$dm = \frac{3M}{\cancel{4\pi}(R_2^3 - R_1^3)} \cancel{4\pi} r^2 dr$$

$$= \frac{3Mr^2 dr}{(R_2^3 - R_1^3)}$$

$$= \frac{2}{3} M \int_{R_1}^{R_2} \frac{3r^2 dr}{(R_2^3 - R_1^3)} r^2 = \frac{M}{(R_2^3 - R_1^3)} \times \frac{2}{\cancel{3}} \cancel{\times 3} \left[\frac{r^5}{5}\right]_{R_1}^{R_2}$$

$$= \frac{2M}{5(R_2^3 - R_1^3)} \left[(R_2^5 - R_1^5)\right]$$

$$= \frac{2}{5} M \left(\frac{(R_2^5 - R_1^5)}{(R_2^3 - R_1^3)} \right)$$

Ans

* **Dynamics of rigid body:-** A rigid body can have following types of motion.

1 Pure Rotation:-

At least one point of object remains at rest.

In this case-

$$K.E \neq \frac{1}{2} m v^2$$

Here, $\boxed{K.E = \frac{1}{2} I \omega^2}$

~~2. Pure Translation:- It means there is no rotation i.e.~~ $\cancel{\boxed{\omega = 0}}$

$$L_{axis} = I_{axis}\,\omega$$

$\boxed{\tau = I_{axis}\,\alpha}$ eq^n of motion.

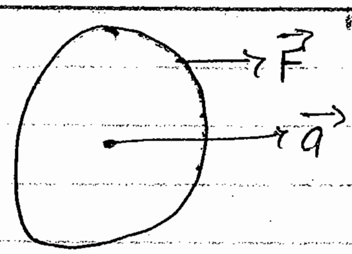
$$\boxed{\frac{dL_{axis}}{dt} = \tau_{axis}}$$

2. Pure Translation:-

It means there is no rotation i.e. $\omega = 0$

$$\boxed{K.E. = \frac{1}{2} m v^2}$$

Equation of motion:-



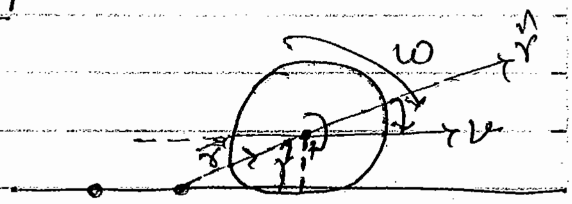
$$\boxed{F_{\text{net force}} = ma}$$

$$\boxed{L = pr_{\perp}}$$

3. General Motion:-

Translation + Rotation

$$\boxed{K.E. = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2}$$

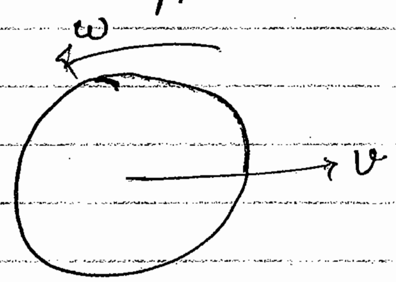


$$\boxed{L = I\omega \pm pr_{\perp}}$$

$r_{\perp} = R$ [If L is calculated about a point on path of round object]

$$\boxed{L = I\omega + pr_{\perp}}$$ { ∵ direction of ω and $r \times v$ is same.

$$\boxed{L = I\omega - pr_{\perp}}$$ { ∵ direction of ω and $r \times v$ is opposite }



$V = \omega r$ → velocity of any particle which is at the distance R at the center. of due to rotation.

---

* **Rolling Motion :-** It is a special case of general motion.

Translational velocity or translational accelaration of center of mass and rotational (Angular) velocity or rotational accelaration about center of mass are related as -

$$\boxed{V_{cm} = \omega R}$$

Diff. w.r. to t.

$$\frac{dv}{dt} = \alpha R$$

→ NET

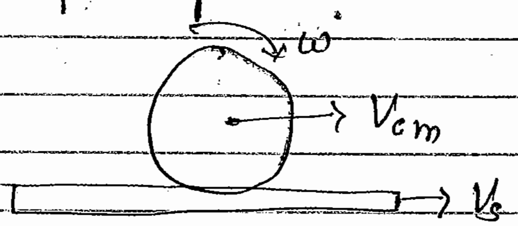
On a stationary surface.

$$V_{cm} = \omega R = V_s$$

$$a_{cm} = \alpha R = a_s$$

→ Grate

On a moving surface.

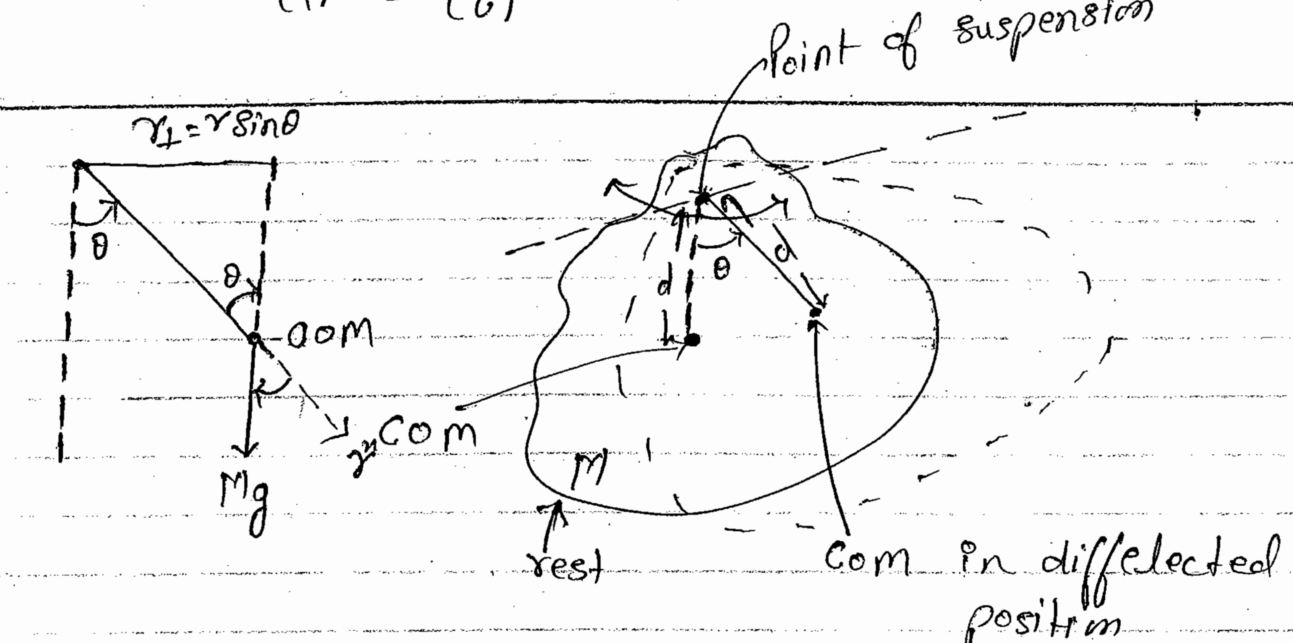


V_{cm} = Velocity of com.

* **Physical Pendulum {Compound Pendulum} :-**

It is a rigid body (rod, disc, ring etc) oscillating about a horizontal axis under the effect of gravity. Time period oscillation.

Direc^n of angular acc^n (α) = direc^n of Torque (τ).



It is a pure rotation case - So Equation of motion.

$$\boxed{\tau = I\alpha}$$

$$\boxed{Mg\,d\sin\theta = I\alpha}$$

I = M.I about horizontal axis through point of suspension.

If θ is small:-

$$\sin\theta \approx \theta$$

$$Mgd\theta = I\alpha$$

$$\boxed{\alpha = \frac{Mgd}{I}\theta}$$

Torque is opposite of θ acc to. direc^n

and $\boxed{\vec{\alpha} = \frac{mgd}{I}(-\vec{\theta})}$

This is the case of simple hormonic motion. Standard eq^n of S.H.M.

$$\alpha = \omega^2 (-\vec{\theta})$$

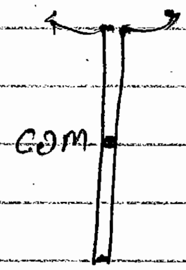
ω ↳ Angular frequency

$$\omega^2 = \frac{mgd}{I}$$

$$\frac{2\pi}{T} = \sqrt{\frac{mgd}{I}}$$

$$\boxed{T = 2\pi\sqrt{\frac{I}{mgd}}}$$

Ques A thin rod of mass m and length L is suspended from 1 m. what is its time period.



$$T = 2\pi\sqrt{\frac{I}{mgd}}$$

$$I = \frac{ML^2}{3}$$

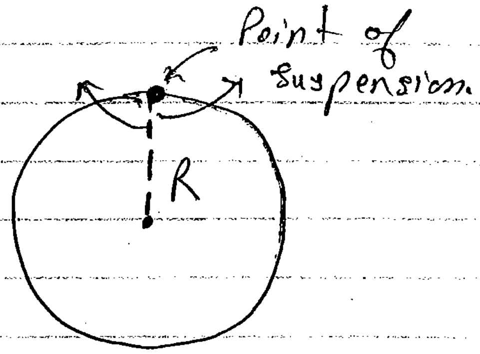
$$d = \frac{L}{2}$$

$$T = 2\pi\sqrt{\frac{ML^2/3}{MgL/2}}$$

$$\boxed{T = 2\pi\sqrt{\frac{2L}{3g}}}$$

Q. A disc of radius R is suspended from a point of its periferi. If T_1 and T_2 be time period of oscillation parallel to and perpendicular to the plane of disc. What is the value of T_1 and T_2?

Sol^n



$\because \quad d = R$

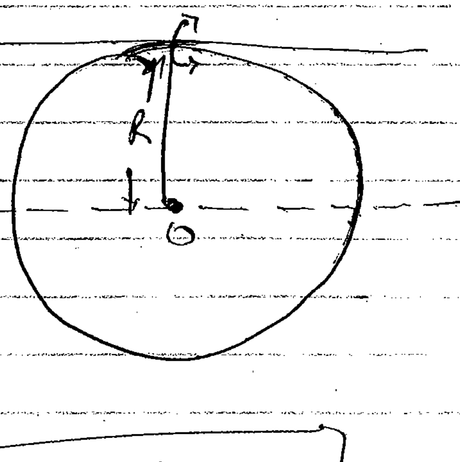
By parallel axis theorem –

$$I = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$$

$$T_1 = 2\pi\sqrt{\frac{I}{Mgd}}$$

$$= 2\pi\sqrt{\frac{\frac{3}{2}MR^2}{M.g.R}}$$

$$\boxed{T_1 = 2\pi\sqrt{\frac{3R}{2g}}}$$



$\dfrac{MR^2}{4}$

$$I = \frac{MR^2}{4} + MR^2$$

$$I = \frac{5}{4}MR^2$$

$$T_2 = 2\pi\sqrt{\frac{I}{Mgd}}$$

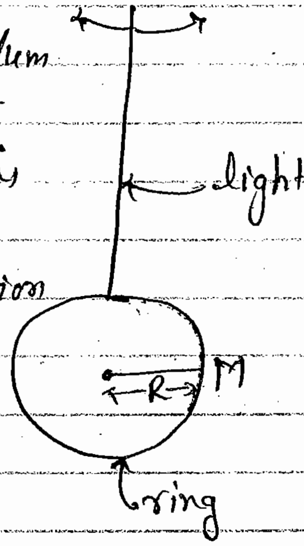
$$T_2 = 2\pi\sqrt{\frac{5MR^2/4}{Mg.R}} \qquad \boxed{T_2 = 2\pi\sqrt{\frac{5R}{4g}}}$$

$$\frac{T_1}{T_2} = \sqrt{\frac{3}{2} \times \frac{4}{5}}$$

$$\boxed{\frac{T_1}{T_2} = \sqrt{\frac{6}{5}}}$$

Yet 203

Q. System oscillates parallel ~~of~~ to plane of ring. What is time period.



It is not a simple pendulum becoz in simple pendulum ~~mass of bod is~~ boob is taken as a particle which not have radius and dimension but here Radius and dimension is

$I = MR^2 + M(R+L)^2$ there.

$d = R + l$

$$T = 2\pi\sqrt{\frac{MR^2 + M(R+L)^2}{Mg(R+l)}}$$ It is compound ~~ple~~ pendulum.

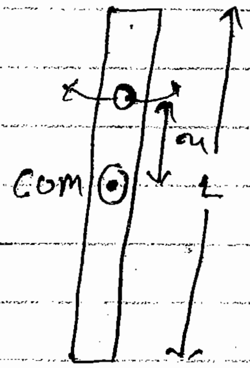
$$= 2\pi\sqrt{\frac{R^2 + R^2 + l^2 + 2Rl}{(R+l)}}$$

$$\boxed{T = 2\pi\sqrt{\frac{2R^2 + l^2 + 2Rl}{(R+l)}}}$$

Q. A thin rod of length L is suspended from some point on its length what should be distance of point of

suspension from COM. So that time period should be minimum.

Sol^n



$d = x$

$$I = \frac{ML^2}{12} + Mx^2$$

$$T = 2\pi\sqrt{\frac{I}{Mgd}}$$

$$= 2\pi\sqrt{\frac{M\left(\frac{L^2}{12} + x^2\right)}{Mgx}}$$

$$T^2 = \frac{4\pi^2}{g}\left(\frac{L^2}{12x} + x\right)$$

If T is minimum then T^2 is also minimum.

$$\frac{dT^2}{dx} = 0$$

$$\frac{4\pi^2}{g}\left(\frac{-L^2}{12x^2} + 1\right) = 0$$

$$-\frac{L^2}{12x^2} + 1 = 0$$

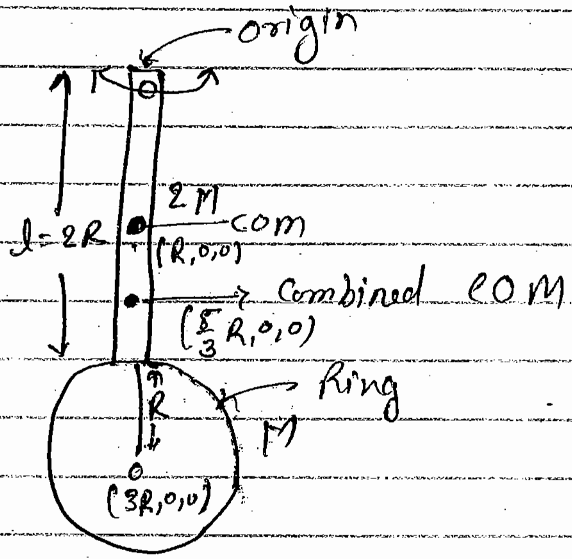
$$x^2 = \frac{L^2}{12}$$

$$\boxed{x = \frac{L}{\sqrt{12}}}$$ Ans

Q. Suppose there is a rod and there is a ring, mass of Rod is 2M and mass of ring is M. What is T for oscillation.

Sol^n

$$T = 2\pi\sqrt{\frac{I_{total}}{M_{total}\, g d}}$$



d = distance of combined COM from point of suspension

$$I = I_c + M(dis)^2$$

$$I_{total} = MR^2 + M(R+l)^2 + \frac{(2M)l^2}{3}$$

Co-ordinate of center of mass :-

$$X_{com} = \frac{\sum m_i x_i}{\sum m_i}$$ } discreate case or for combination of object.

$$X_{com} = \frac{\int x\, dm}{\int dm}$$ } for a single continuous object.

In this case -

$$X_{com} = \frac{2M \times R + M \times 3R}{3M}$$

$$X_{com} = \frac{5}{3}R = d$$

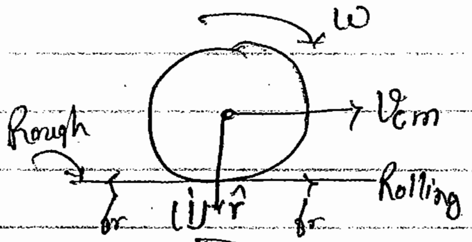
$$T = 2\pi\sqrt{\frac{\cancel{M}R^2 + \cancel{M}(R+l)^2 + (2\cancel{M})l^2/3}{\cancel{3}\cancel{M}\cdot g\cdot \frac{5}{\cancel{3}}R}}$$

$$= 2\pi\sqrt{\frac{R^2 + R^2 + l^2 + 2Rl + 2l^2/3}{5gR}}$$

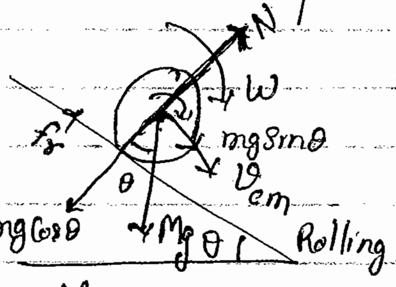
$$= 2\pi\sqrt{\frac{6R^2 + 3l^2 + 2Rl + 2l^2}{15gR}}$$

$$\boxed{T = 2\pi\sqrt{\frac{6R^2 + 5l^2 + 2Rl}{15gR}}}$$ Ans

* Rolling Motion on Stationary Surface (horizontal or inclined) :-

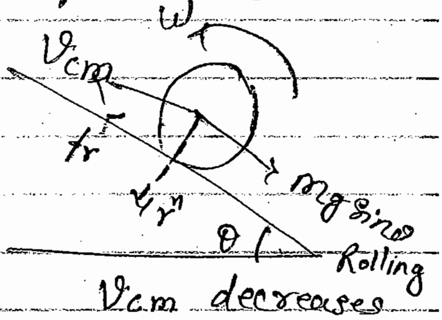


(i)



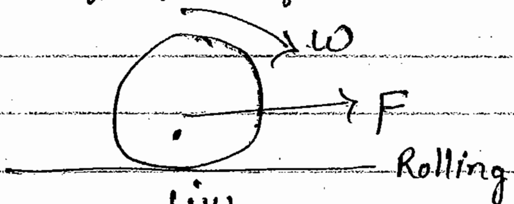
v_{cm} increases due to mg sinθ.

(ii)



v_{cm} decreases mg sinθ.

(iii)



(iv)

Condition for Rolling :-

$$v_{cm} = \omega R$$

$$a_{cm} = \alpha R$$

from fig (ii) :-

→ Here v_{cm} increases due to Mg sinθ

$\therefore \quad v_{cm} = \omega R$

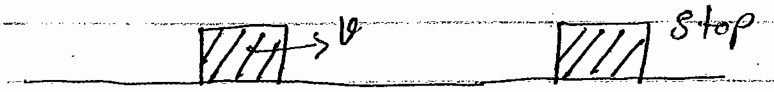
so due to this relation ω should also be increases but, Here torque is zero

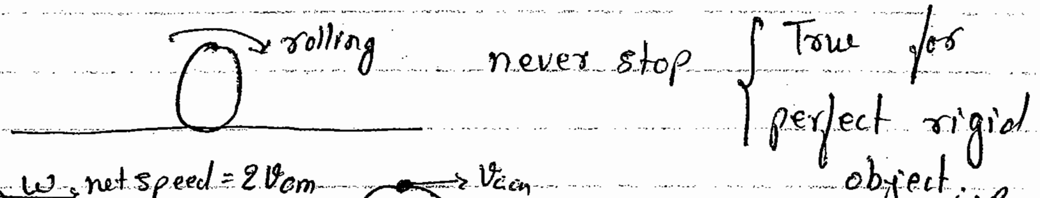
because Mg and N both are passing through center.
So Here ω is not increases so this Rolling is not possible.
So Rolling is not possible on smooth inclined plane.
for Rolling motion on incliened plane friction must be present.
When incliened plane is a rough surface then friction is works here.

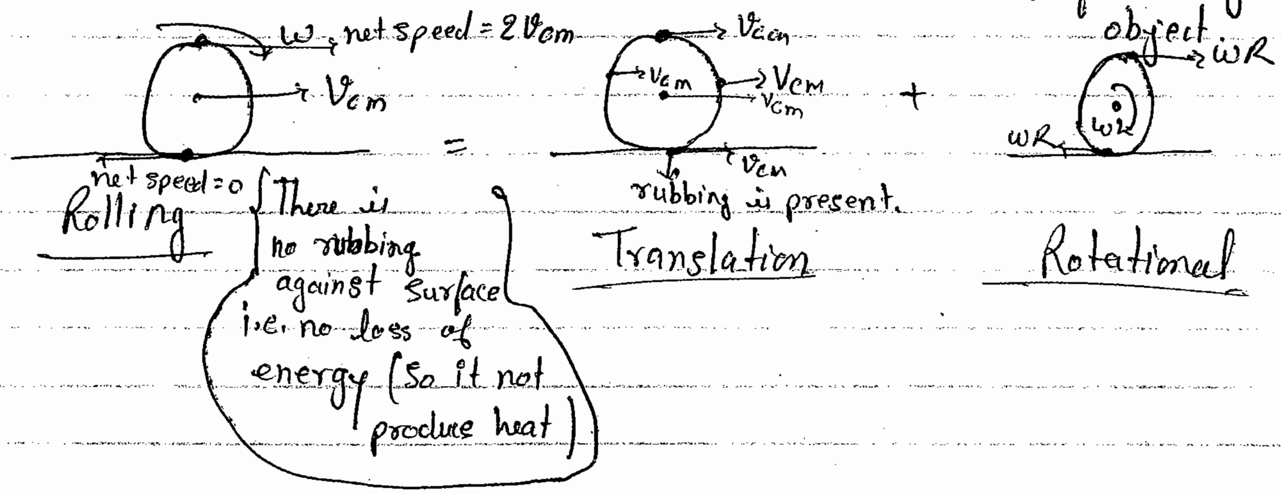
(i) from fig. (i) :-
In this case when friction works opposite to the motion then v_{cm} decreases but $r \times F$ and ω are in same direcn then ω is increases so this can not satisfies the condition of Rolling.
In other hand when friction in the direcn of motion then v_{cm} increases but $r \times F$ and ω is in opposite direction so ω is decreases so it is also not satisfies the condition of rolling.
So here for Rolling motion surface might be smooth then $v_{cm} = $ constt. and $\omega = $ constt.

refectely Rigit that means we can't deform.
Iron are not ~~ref~~ perfectily rigid.

* In rolling motion although surface might be rough there is no loss of energy against ~~further~~ friction.



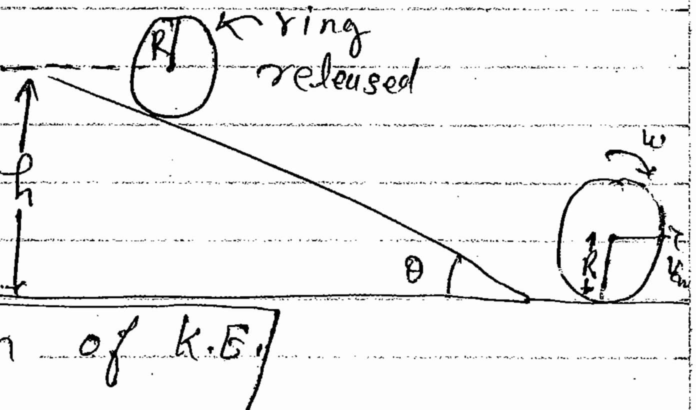




Net 2013 June.

Q. A ring is released from a inclined plane whose center is at h distance from ground. If ring rolls down the plane what is its angular speed when it reaches the bottom.

Sol^n



Here center decreases by (h-R) height.

$v_{cm} = \omega R$

Apply Conservation of energy :-

Loss of P.E. = Gain of K.E.

$$Mg(h-R) = \frac{1}{2} I\omega^2 + \frac{1}{2} M v_{cm}^2$$

$$= \frac{1}{2} MR^2\omega^2 + \frac{1}{2} M\omega^2 R^2$$

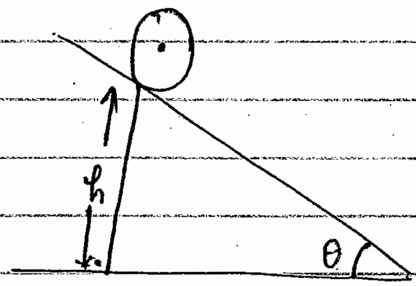
$$\cancel{M}g(h-R) = \cancel{M}R^2\omega^2$$

$$\boxed{\omega = \sqrt{\frac{g(h-R)}{R^2}}}$$

Q. A ~~hollow~~ solid sphere and a disc rolls down from an inclined plane from same point what is ratio of velocity of sphere and velocity of disc when they reach at the bottom.

Sol^n

for Disc :-



By Conservation of energy :-

$$\boxed{\text{Loss in P.E.} = \text{Gain in K.E.}}$$

$$mgh = \frac{1}{2} I\omega^2 + \frac{1}{2} m v_{cm}^2 \qquad \left\{ v_{cm} = \omega r \right.$$

$$= \frac{1}{2} \frac{m\cancel{r^2}}{2} \cdot \frac{v_{cm}^2}{\cancel{r^2}} + \frac{1}{2} m v_{cm}^2$$

$$\cancel{m}gh = \frac{3}{4} \cancel{m} v_{cm}^2$$

$$\boxed{(v_{cm})_{disc} = \sqrt{\frac{4}{3} gh}}$$

$s = ut + \frac{1}{2}at^2$ when $a = 0$ then $s = ut$

and $\boxed{t = \frac{s}{u}}$ only when u = Constant

$\text{time} = \frac{\text{distance}}{\text{Velocity}}$

for sphere :-

solid sphere

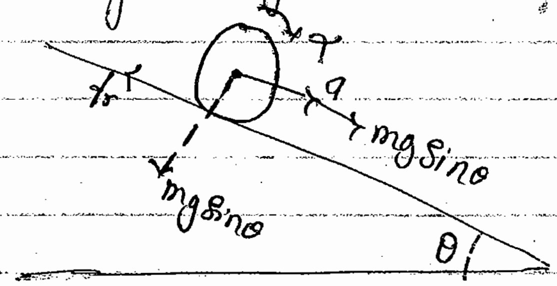
$$Mgh = \frac{1}{2}\left(\frac{2}{5}MR^2\right)\cdot\frac{V_{cm}^2}{R^2} + \frac{1}{2}MV_{cm}^2$$

$$\cancel{M}gh = \frac{7}{10}\cancel{M}V_{cm}^2$$

$$\boxed{(V_{cm})_{sphere} = \sqrt{\frac{10}{7}gh}}$$

$$\boxed{\frac{(V_{cm})_{sphere}}{(V_{cm})_{disc}} = \sqrt{\frac{15}{14}}}$$

* Acceleration of Rolling object on inclined plane :-



Equation of motion for Translation :-

$$mg\sin\theta - f_r = ma \quad \text{——} \ (1)$$

for Rotation :-

$$\tau = I\alpha$$

$$f_r R = I\alpha$$

for Rolling :-

$$a = \alpha R$$

height h ... s distance.

$$\boxed{\alpha = \frac{a}{R}}$$

$$f_r R = \frac{Ia}{R}$$

$$\boxed{f_r = \frac{Ia}{R^2}} \leftarrow (ii)$$

(i) + (ii) :-

$$mg\sin\theta = a\left[m + \frac{I}{R^2}\right]$$

$$g\sin\theta = a\left[1 + \frac{I}{mR^2}\right]$$

$$\boxed{a = \frac{g\sin\theta}{1 + \frac{I}{mR^2}}}$$

for friction put 'a' in eq^n (ii).

A-7

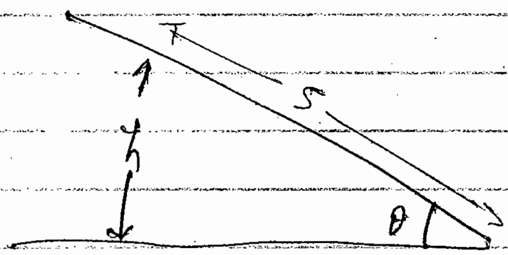
Q.5 A solid cylinder starts Rolling down an inclined plane of inclination θ, from height h. Time taken to reach the bottom is ?.

(a) $\sqrt{\frac{2h}{g}}$ (b) $\sqrt{\frac{3h}{g}}$ (c) $\frac{1}{\sin\theta}\sqrt{\frac{3h}{2g}}$

✓(d) $\frac{1}{\sin\theta}\sqrt{\frac{3h}{g}}$

Sol^n

$$s = \frac{h}{\sin\theta}$$



$$s = ut + \frac{1}{2}at^2$$

$$s = 0 + \frac{1}{2}at^2$$

$$\boxed{t = \sqrt{\frac{2s}{a}}} \text{ — (a)}$$

$\left\{ \because a = \dfrac{g\sin\theta}{1+\frac{I}{mR^2}} \right\}$

$$a = \frac{g\sin\theta}{1+\frac{1}{2}} = \frac{2g\sin\theta}{3}$$

$\left\{ \because \text{M.I. of } \cancel{\text{Sphere}} \text{ cylinder} = \frac{1}{2}MR^2 \right.$

$$t = \sqrt{\cancel{2}\frac{h}{\sin\theta} \cdot \frac{3}{\cancel{2}g\sin\theta}}$$

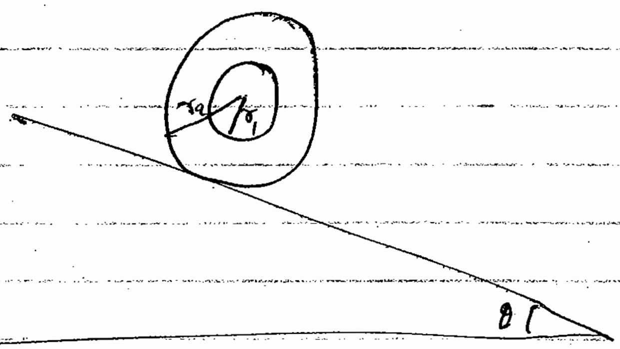
$$\boxed{t = \frac{1}{\sin\theta}\sqrt{\frac{3h}{g}}}$$

A-7

Q.15 A circular loop of inner and outer radii r_1 and r_2 rolls down an inclined plane of inclination θ. Angular acceleration of loop on the inclined plane is - ?

(a) $\dfrac{2r_2}{3r_2^2+r_1^2}g\sin\theta$ (b) $\dfrac{2r_1}{3r_2^2+r_2^2}g\sin\theta$ (c) $\dfrac{r_2}{r_2^2+r_1^2}g\sin\theta$

(d) $\dfrac{r_1}{r_2^2+r_1^2}g\sin\theta$.



Sol $a = \dfrac{g\sin\theta}{1+\frac{I}{mr_2^2}}$

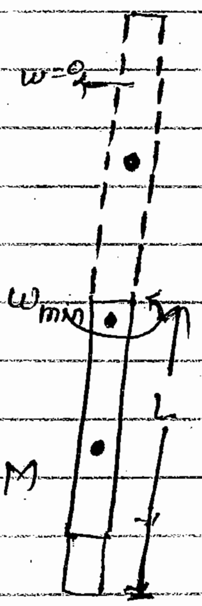
$$\boxed{a = \alpha r_2}$$

$$\boxed{a = \frac{g \sin\theta}{1 + \frac{1}{2}\frac{m(r_1^2+r_2^2)}{m r_2^2}}}$$

$$\alpha = \frac{2 r_2^{\cancel{2}} g \sin\theta}{2r_2^2 + (r_1^2+r_2^2) \times \cancel{r_2}}$$

$$\boxed{\alpha = \frac{2 r_2 g \sin\theta}{r_1^2 + 3r_2^2}}$$ Ans

A-7
Q.6



Since here one point is fixed so it is a case of pure rotation.

Loss in K.E. = Gain in P.E.

$$\frac{1}{2} I \omega_{min}^2 - 0 = mgL$$ { here L = height accended by center

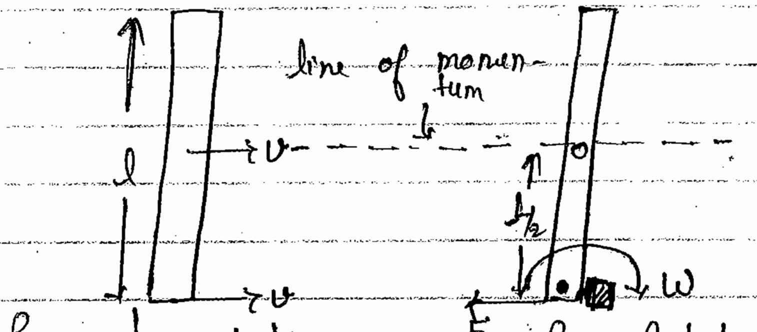
$$\Rightarrow \frac{1}{2}\frac{\cancel{M}L^{\cancel{2}}}{3}\omega_{min}^2 = \cancel{m}g\cancel{L}$$

$$\boxed{\omega_{min} = \sqrt{\frac{6g}{L}}}$$

A-7

Q.7 A rod of length 'L'

Sol^n If initially no point is fixed but of towards one point fixed then conservation of energy will not be applied. because it is like perfectly inelastic collesion.



Pure translation — Pure Rotation

$$\tau = \frac{dL}{dt}$$

If $\tau = 0$

$$\frac{dL}{dt} = 0$$

L = constant

Apply Conservation of angular momentum:- {about the point which is fixed}.

$$L_i = L_f$$

$$p r_\perp = I\omega$$

$$\cancel{m} v \frac{\cancel{l}}{2} = \frac{\cancel{m} l^{\cancel{2}}}{3}\omega$$

$$\boxed{\omega = \frac{3v}{2l}}$$

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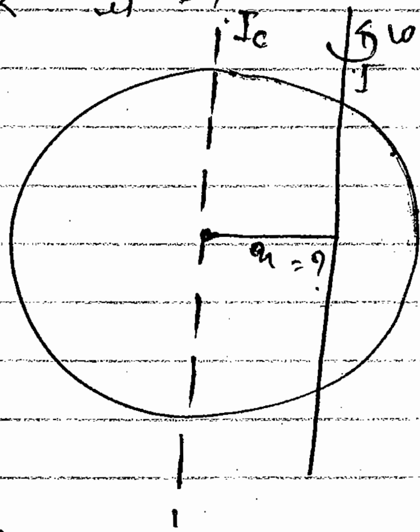
A-7

Q.3 A circular disc of mass M and Radius R rotates about an axis lying in its plane with constant angular velocity ω. If angular momentum of the disc about this axis is $\frac{3}{4}MR^2\omega$, distance of axis from centre of disk is -?

Sol^n



Given $L_{axis} = \frac{3}{4}MR^2\omega$

$$I_{axis}\,\omega = \frac{3}{4}MR^2\omega$$

$$(I_C + M\cancel{R}^2)\,\omega = \frac{3}{4}MR^2\omega$$

$$\because\ I_C = \frac{MR^2}{4}$$

$$\left(\frac{MR^2}{4} + Mx^2\right)\cancel{\omega} = \frac{3}{4}MR^2\cancel{\omega}$$

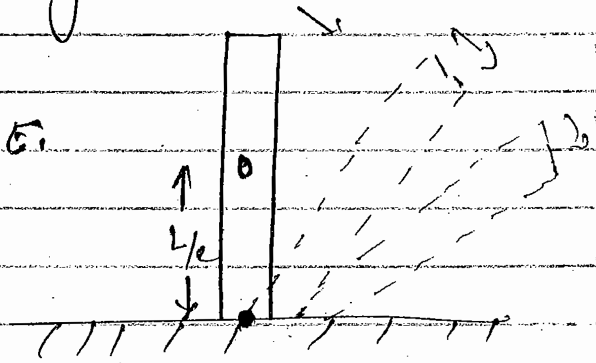
$$Mx^2 = \frac{1}{2}MR^2$$

$$\boxed{x = \frac{R}{\sqrt{2}}}\ \underline{\text{Ans}}$$

A-7

Q.8 A Rod of length L is held vertical with its one end fixed on ground. The rod can rotate about fixed end. If the rod is released from this position, what is its speed when it is about to hit the ground.

Sol^n Conservation of energy



Loss in P.E. = Gain in K.E.

$$Mg\frac{L}{2} = \frac{1}{2}I\omega^2$$

$$\cancel{M}g\frac{L}{\cancel{2}} = \frac{1}{\cancel{2}}\frac{\cancel{M}L^{\cancel{2}}}{3}\omega^2$$

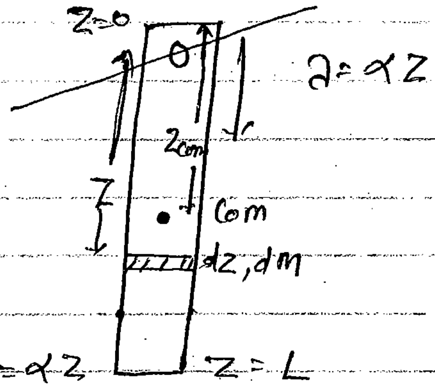
$$\omega^2 = \frac{3g}{L}$$

$$\boxed{\omega = \sqrt{\frac{3g}{L}}}$$ Angular speed.

A-7
Q.10 A thin rod of length L is suspended from one end as shown in figure. The linear mass density of rod varies with distance from point of suspension as $\lambda = \alpha s$. Time period of oscillation of the rod is ?

Sol^n

$$T = 2\pi\sqrt{\frac{I}{Mgd}}$$



I = MI about point of susp.
d = dist of (P.S.) from C.O.M.

$$I = \int dm\, r_\perp^2 \qquad \lambda = \frac{dm}{dz} = \alpha z$$

$$dm = \alpha z\, dz$$

$$= \int_0^L \alpha z\, dz\, z^2$$

$$\boxed{M.I. = \frac{\alpha L^4}{4}}$$

Position of center of mass -

$$z_{com} = \frac{\int z\, dm}{\int dm} = \frac{\int z\, \alpha z\, dz}{\int \alpha z\, dz} = \frac{\int z^2 \alpha\, dz}{M}$$

$$= \frac{\alpha L^3}{3M} = d$$

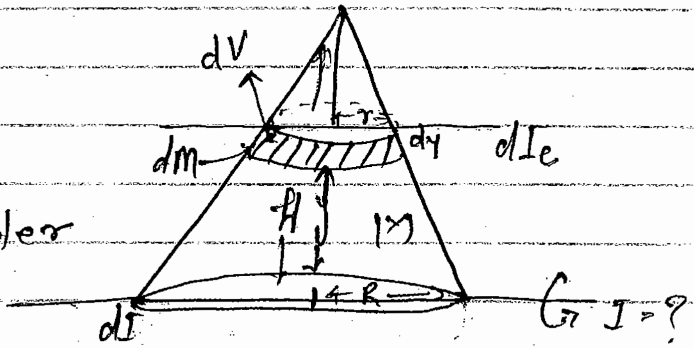
So $$T = 2\pi\sqrt{\frac{\frac{\alpha L^4}{4}}{\cancel{M} g\left(\frac{L^3\alpha}{3\cancel{M}}\right)}}$$ $$\boxed{T = 2\pi\sqrt{\frac{3L}{4g}}}$$ Ans

A-7

Q.19 A right circular cone has mass M, Radius R and height h. Moment of inertia of cone about a diameter of its base is?

Sol^n

$$I = \int dI$$



M.I of disc about the given axis (base diameter

$$dI = dI_c + dM y^2$$

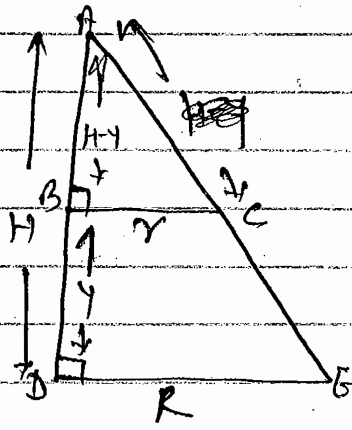
$$= \frac{dM r^2}{4} + dM y^2$$

$$\boxed{dI = dM\left(\frac{r^2}{4} + y^2\right)} \quad \text{—— (I)}$$

$\because \quad \rho = \frac{M}{\frac{1}{3}\pi R^2 H} = \frac{dM}{dV}$

$$dM = \frac{M}{\frac{1}{3}\pi R^2 H} dV$$

$$\boxed{dM = \frac{3M}{\pi R^2 H} \pi r^2 dy} \quad \text{—— (II)}$$



$\because$ Here ΔABC and ΔADG is similar so-

$$\frac{r}{R} = \frac{H-y}{H}$$

$$\boxed{r = R\left(1 - \frac{y}{H}\right)} \quad \text{—— (III)}$$

$$\boxed{y = H\left(1 - \frac{r}{R}\right)}$$

$$dr = \frac{-R}{H}\,dy$$

So $\boxed{dy = -\frac{H}{R}\,dr}$

So from (I)

$$dI = \frac{3M}{\cancel{\pi} R^2 H}\,\cancel{\pi} r^2 \cdot \left(\frac{-H}{R}dr\right)\left(\frac{r^2}{4} + H^2\left(1-\frac{r}{R}\right)^2\right)$$

$$\cancel{dI = \frac{3M}{R^3}\left[\frac{r^4}{4}dr + H r^2 dr - \frac{H}{R} r^3 dr\right]}$$

$$= \frac{3M}{R^3} r^2 dr\left[\frac{r^2}{4} + H^2\left(1 + \frac{r^2}{R^2} - 2\frac{r}{R}\right)\right]$$

$$\int dI = \frac{3M}{R^3}\left[\int_0^R \frac{r^4}{4}dr + H^2\left(\int_0^R r^2 dr + \int_0^R \frac{r^4}{R^2}dr - \frac{2}{R}\int_0^R r^3 dr\right)\right]$$

$$= \frac{3M}{R^3}\left[\frac{1}{4}\frac{R^5}{5} + H^2\left(\frac{R^3}{3} + \frac{1}{R^2}\left(\frac{R^5}{5}\right) - \frac{2}{R}\left(\frac{R^4}{4}\right)\right)\right]$$

$$= \frac{3M}{R^3}\left[\frac{R^5}{20} + H^2R^3\left[\frac{1}{3} + \frac{1}{5} - \frac{1}{2}\right]\right]$$

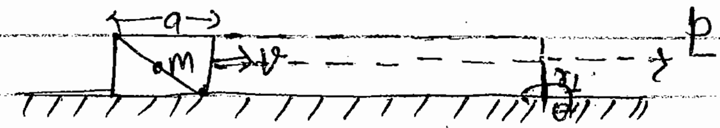
$$= \frac{3MR^2}{20} + 3MH^2\left(\frac{10+6-15}{30}\right)$$

$$= \frac{3MR^2}{20} + \frac{\cancel{3}MH^2}{\cancel{30}\,10}$$

$$\boxed{I = \frac{3MR^2}{20} + \frac{MH^2}{10}}$$ Ans

A-7

Q.24 A cubical block of side 'a' is moving with velocity 'v' on a horizontal smooth plan as shown. It hits a ridge at point O. The angular speed of the block after it hits O is?

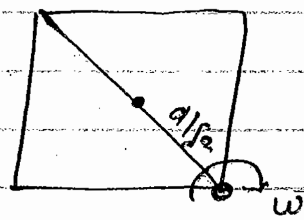


Solⁿ So applying Conservation of Angular momentum -

$L_i = L_f$

$p r_\perp = I\omega$

$M v \cdot \frac{a}{2} = I\omega$



$\frac{Ma^2}{2}$

So $Mv \cdot \frac{a}{2} = \left[\frac{Ma^2}{6} + \frac{Ma^2}{2}\right]\omega$

$\Rightarrow \cancel{M} v \cdot \frac{\cancel{a}}{2} = \cancel{M} a^{\cancel{2}}\omega\left[\frac{1}{6}+\frac{1}{2}\right]$

$\Rightarrow \frac{v}{2} = \omega a\left[\frac{1+3}{6}\right]$

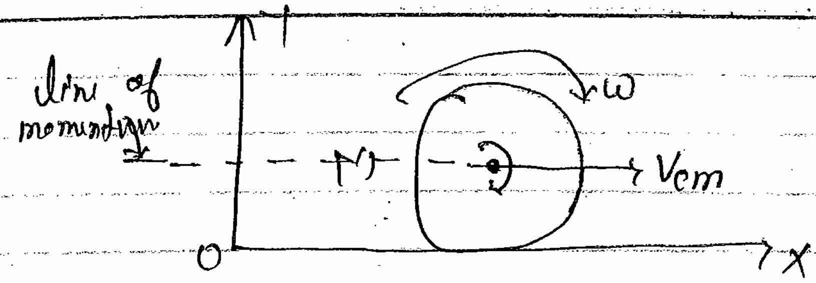
$\Rightarrow \frac{v}{\cancel{2}} = \frac{4\omega a}{\cancel{6}\,3}$

$\Rightarrow \boxed{\omega = \frac{3v}{4a}}$ So ans (a) is correct.

A-7

Q.23 A disc of mass M and Radius R is rolling with angular speed ω on a horizontal plan as shown in figure. The magnitude of angular momentum of the disc about the origin O is?

Solⁿ



$$L = I\omega + M V_{cm} R$$

$$= \frac{MR^2}{2}\omega + M\omega R.R$$

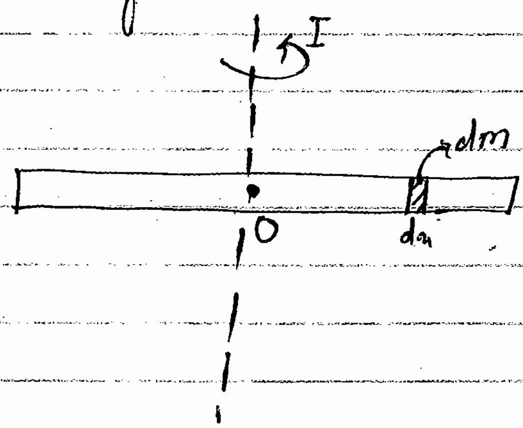
$$= \frac{MR^2\omega}{2} + M\omega R^2$$

$$\boxed{L = \frac{3}{2} M\omega R^2}$$

A-6 Q.9 modified

Q. Calculate moment of Inertia about a perpendicular axis through its center of mass.

$\lambda = \lambda_0 |x|$ $\quad x =$ dist. from center



$$\lambda = \frac{dm}{dx}$$

$$\lambda_0 |x| = \frac{dm}{dx}$$

$$dm = \lambda_0 |x| dx$$

$$I = \int dm\, x^2$$

$$= \int_{-l}^{+l} \lambda_0 |x| dx . x^2$$

$$= \int_{-l}^{0} \lambda_0 (-x)\, x^2 dx + \int_{0}^{l} \lambda_0 (x)\, x^2 dx$$

$$I = -\lambda_0 \left[\frac{x^4}{4}\right]_{-l}^{0} + \lambda_0 \left[\frac{x^4}{4}\right]_{0}^{l}$$

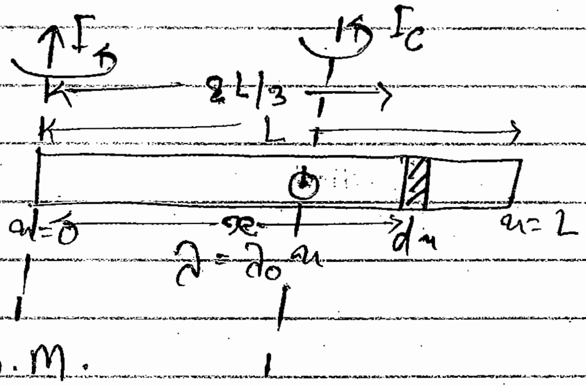
$$= +\lambda_0 \frac{l^4}{4} + \lambda_0 \frac{l^4}{4} = \frac{2\lambda_0 l^4}{4}$$

$$\boxed{I = \frac{\lambda_0 l^4}{2}}$$

$$x_{com} = \frac{\int x\,dm}{\int dm} = \frac{\int_{-L}^{+L} x \lambda_0 |x|\,dx}{\int_{-L}^{+L} \lambda_0 |x|\,dx} = 0$$

$$\boxed{x_{com} = 0}$$

Q. When x is measured from one end. What is the M.I of Rod about a $\perp^{r}$ axis through its c.o.m.



Sol^n

$$\lambda = \frac{dm}{dx} = \lambda_0 x$$

$$dm = \lambda_0 x\,dx$$

$$I = \int dm\, r_\perp^2$$

$$\because r_\perp = x$$

$$\therefore I = \int_0^L \lambda_0 x\,dx \cdot x^2$$

$$\boxed{I = \frac{\lambda_0 L^4}{4}}$$

$$X_{com} = \frac{\int x\,dm}{\int dm} = \frac{\int_0^{l} x\cdot\lambda_0 x\,dx}{\int_0^{l} \lambda_0 x\,dx}$$

$$= \frac{\lambda_0 \frac{L^3}{3}}{\lambda_0 \frac{L^2}{2}} \rightarrow M$$

$$\boxed{X_{com} = \frac{2L}{3}}$$

So $$I = I_c + M\left(\frac{2L}{3}\right)^2$$

$$\frac{\lambda_0 L^4}{4} = I_c + \frac{\lambda_0 L^2}{2}\cdot\frac{4L^2}{9}$$

$$\frac{\lambda_0 L^4}{4} = I_c + \frac{2\lambda_0 L^4}{9}$$

$$I_c = \lambda_0 L^4\left(\frac{1}{4} - \frac{2}{9}\right) = \lambda_0 L^4\left(\frac{9-8}{36}\right)$$

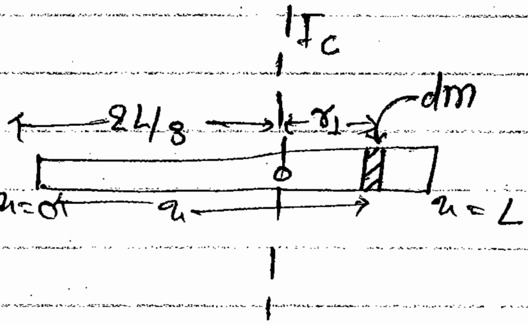
$$\boxed{I_c = \frac{\lambda_0 L^4}{36}}$$ Ans

Method-II :- Giving I_c directely :-

first calculate ~~c.o.~~ position of com.

$$I = \int dm\, r_1^2$$

$$= \int_0^{L} \lambda_0 x\,dx\left(x - \frac{2L}{3}\right)^2$$



$$r_1 = x - \frac{2L}{3}$$

$$I = \lambda_0 \int_0^L x\left(x^2 + \frac{4}{9}L^2 - \frac{4}{3}Lx\right)dx$$

$$= \lambda_0\left[\int_0^L x^3\,dx + \frac{4L^2}{9}\int_0^L x\,dx - \frac{4}{3}L\int_0^L x^2\,dx\right]$$

$$= \lambda_0\left[\frac{L^4}{4} + \frac{\cancel{4}\,2L^2}{9}\cdot\frac{L^2}{\cancel{2}} - \frac{4}{3}L\,\frac{L^3}{3}\right]$$

$$= \lambda_0\left[\frac{L^4}{4} + \frac{2L^4}{9} - \frac{4L^4}{9}\right]$$

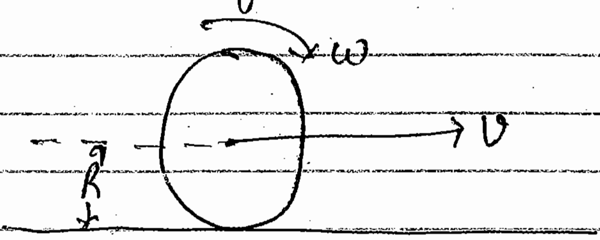
$$= \lambda_0 L^4\left[\frac{1}{4} - \frac{2}{9}\right] = \lambda_0 L^4\left[\frac{9-8}{36}\right]$$

$$I = \frac{1}{36}\lambda_0 L^4$$

So $\boxed{I = \dfrac{\lambda_0 L^4}{36}}$ Ans

Q. A circular disc of mass M and radius R is moving on a horizontal surface with speed v if angular momentum of the disc about the point on its path is $2MVR$. what is K.E. of the disc?

Sol^n



$$L = I\omega + mvR$$

$$2mvR = I\omega + mvR$$

$$mvR = I\omega$$

$$\cancel{m}v\cancel{R} = \frac{\cancel{m}R^{\cancel{2}}}{2}\omega \Rightarrow \boxed{\omega = \frac{2v}{R}}$$

$$K.E. = \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$$

$$= \frac{1}{\cancel{2}} \frac{m\cancel{R^2}}{\cancel{2}} \cdot \frac{\cancel{4}v^2}{\cancel{R^2}} + \frac{1}{2} mv^2$$

$$\boxed{K.E. = \frac{3}{2} mv^2}$$

* Pure Rotational Motion of Rigid Body :- At least one point of rigid body is fixed.

It has two types -

(I) Rotation about a fixed axis (Here point is also fixed)

(II) Rotation about a fixed point (axis is not fixed.).

Case I :-

$L_{axis} = I\omega$ → This may or may not be total angular momentum.

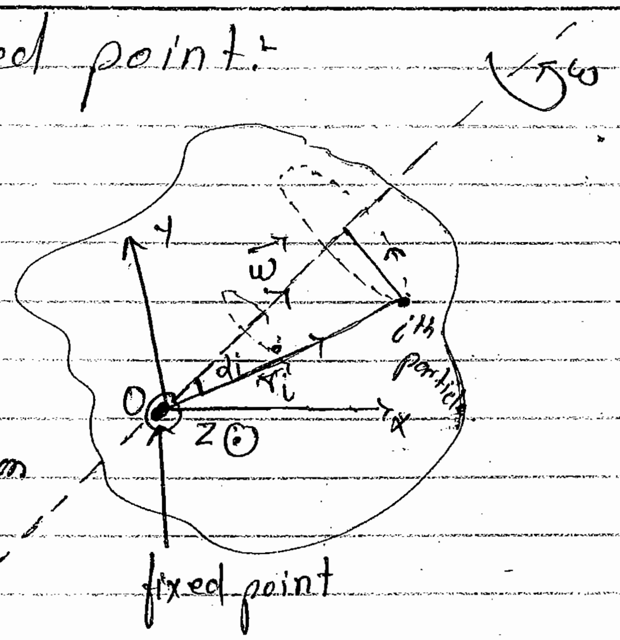
$$\boxed{K.E. = \frac{1}{2} I\omega^2}$$

Case II :-

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$$

$$\boxed{K.E. = \frac{1}{2} \vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega}}$$

* Rotation about a fixed point:-



Radius of circul in which i^{th} particle moves is :- $r_i \sin\alpha_i$

tangential velocity of i^{th} particle due to rotation

$v_i = \omega r \to$ Radius

So speed of i^{th} particle $\to$ Axis about which the rigid body is ~~osci~~ rotating at semi instant of time.

$$v_i = \omega r_i \sin\alpha_i$$

$$\boxed{\vec{v_i} = \vec{\omega} \times \vec{r_i}}$$

{ We can't write $\vec{r_i} \times \vec{\omega}$ becoz it can't give correct sence of velocity.

Angular momentum of i^{th} particle about point 'O'

$$\vec{L}_{i^{th}} = \vec{r_i} \times \vec{p_i}$$

$$= \vec{r_i} \times m_i \vec{v_i}$$

$$= m_i(\vec{r_i} \times \vec{v_i}) = m_i[\vec{r_i} \times (\vec{\omega} \times \vec{r_i})]$$

Angular momentum of rigid body about point O.

$$\vec{L} = \sum_{i=1}^{N} \vec{L_i}$$

$$\boxed{\vec{L} = \sum_{i=1}^{N} m_i[\vec{r_i} \times (\vec{\omega} \times \vec{r_i})]}$$

$$\vec{r_i} = x_i\hat{x} + y_i\hat{y} + z_i\hat{z}$$

$$\vec{\omega} = \omega_x\hat{x} + \omega_y\hat{y} + \omega_z\hat{z}$$

$$\vec{L} = L_x\hat{x} + L_y\hat{y} + L_z\hat{z}$$

Put $\vec{r_i}$, $\vec{\omega_i}$ and $\vec{L}$ in above expression to get [equate coefficients of $\hat{x}$, $\hat{y}$, and $\hat{z}$)

$$L_x = \underbrace{\sum m_i(y_i^2 + z_i^2)}_{I_{xx}}\omega_x - \underbrace{\sum m_i x_i y_i}_{I_{xy}}\omega_y - \underbrace{\sum m_i x_i z_i}_{I_{xz}}\omega_z$$

$$L_y = -\underbrace{\sum m_i y_i x_i}_{I_{yx}}\omega_x + \underbrace{\sum m_i(x_i^2 + z_i^2)}_{I_{yy}}\omega_y - \underbrace{\sum m_i y_i z_i}_{I_{yz}}\omega_z$$

$$L_z = -\underbrace{\sum m_i z_i x_i}_{I_{zx}}\omega_x - \underbrace{\sum m_i z_i y_i}_{I_{zy}}\omega_y - \underbrace{\sum m_i(x_i^2 + y_i^2)}_{I_{zz}}\omega_z$$

So. $L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \underbrace{\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}}_{\text{Co-ordinate dependent}} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

values of elements will change if axis chosen are changed.

Here the axes is choosen in that manner that off diagonal elements will be zero.

- Here matrix (3×3) is II^{nd} rank Tensor.
- It is symmetric and real.
- So it is a Hermitian matrix.
- its eigen values are real.
- This matrix is known as Inertia Tensor.

So Symbolic representation:-

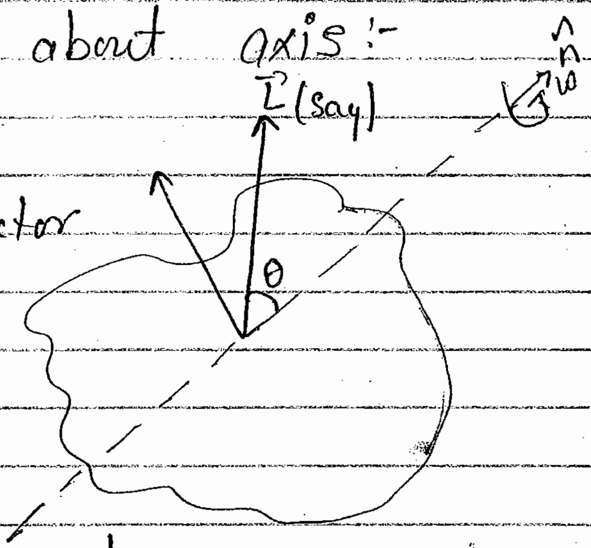
$$\boxed{\vec{L} = \overset{\leftrightarrow}{\mathrm{II}}\cdot\vec{\omega}}$$

Compact form of matrix relation.

Symbol for Inertia Tensor.

It is the general formula for Angular momentum.

* Angular Momentum about axis:-



Let $\hat{n}$ is unit vector along the axis

$$\vec{\omega} = \omega\hat{n}$$

$$L_{axis} = \vec{L}\cdot\hat{n}$$

$$= \hat{n}\cdot\vec{L}$$

$$= \hat{n}\cdot\overset{\leftrightarrow}{\mathrm{II}}\cdot\vec{\omega}$$

$$= \left(\hat{n}\cdot\overset{\leftrightarrow}{\mathrm{II}}\cdot\hat{n}\right)\vec{\omega}$$

$L_{axis} = L\cos\theta$

we know that -

$$L_{axis} = I\omega$$

So $\boxed{I = \hat{n}\cdot\overset{\leftrightarrow}{\mathrm{II}}\cdot\hat{n}}$ (Memorize)

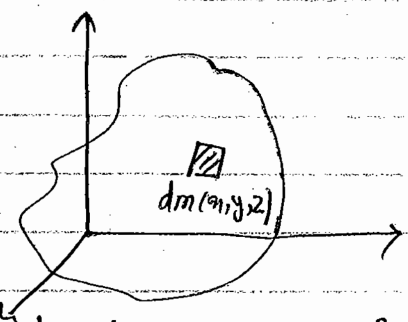
Row Matrix

3x3 Matrix

Coloumn matrix.

* Inertia Tensor :-

$$\mathbb{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$



$I_{xx}, I_{yy}, I_{zz} \longrightarrow$ M.I. about x, y, z axis

$I_{ij}, \; i \neq j \longrightarrow$ Product of inertia.

$$I_{xx} = \sum_{i=1}^{N} m_i (y_i^2 + z_i^2) \quad , \quad I_{xx} = \int dm\,(y^2+z^2)$$

(x, y, z Co-ordinates of dm)

discreate   Continuous

$$I_{xy} = -\sum_{i=1}^{N} m_i x_i y_i \quad , \quad I_{xy} = -\int dm\, xy$$

* Principal Axes :-

The set of three axes for which product of inertia are zero, are called principal Axes [Mathematical Defination].

If x, y, z are principal axes –

$I_{ij} = 0 \quad i \neq j \quad , \quad I_{xy} = 0 \, , \; I_{yz} = 0$ etc.

$$\mathbb{I} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

$\longrightarrow$ Principal Moment of Inertia.

$$\boxed{\text{Principal M.I} = \text{Egen Value of Inertia Tensor}}$$

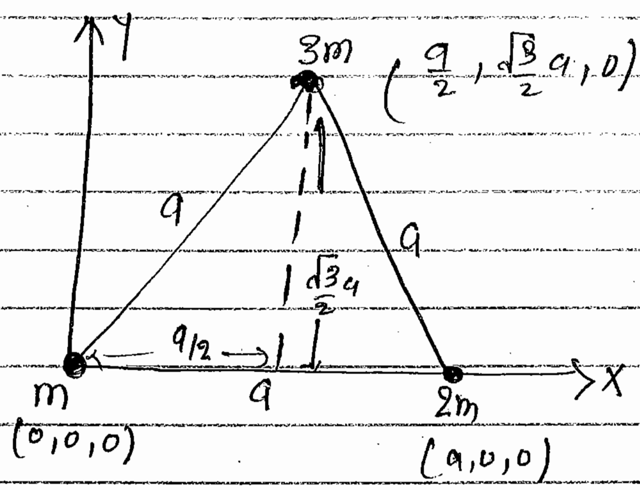
⇒ Direction of Principal Axes :-

Direction of principal Axes are given by eigen vectors of inertia tensor.

A-6
Q.18 In the figure shown value of I_{xy} is ?

$$I_{xy} = -\sum_{i=1}^{3} m_i x_i y_i$$

$$= -\left(m \times 0 + 2m \times 0 + 3m \times \frac{a}{2} \times \frac{\sqrt{3}a}{2}\right)$$



$$\boxed{I_{xy} = -\frac{3\sqrt{3}ma^2}{4}}$$

∵ $I_{xy} \neq 0$ (Product of Inertia $\neq 0$)

So X, Y, Z will ~~be~~ not be principal axes

* Draw the principal Axes about (0,0,0) :-

If We want to find orientation of principal axis then we need to write inertia tensor and find its eigen vectors.

$\therefore I_{yx} = I_{xy} = \frac{-3\sqrt{3}a}{4}$

$I_{yz} = I_{zy} = -\sum m_i y_i z_i = 0$

$I_{zx} = I_{xz} = 0$

$I_{xx} = \sum m_i (y_i^2 + z_i^2) = m \times 0 + 2m \times 0 + 3m\left(\frac{3a^2}{4} + 0^2\right)$

$I_{xx} = \frac{9a^2}{4}$

$I_{yy} = \sum m_i (x_i^2 + z_i^2) = m \times 0 + 2m(a^2 + 0) + 3m\left(\frac{a^2}{4} + 0\right)$

$= 2ma^2 + \frac{3ma^2}{4} = \frac{11ma^2}{4}$

$I_{zz} = \sum m_i (x_i^2 + y_i^2) = m \times 0 + 2m(a^2 + 0) + 3m\left(\frac{a^2}{4} + \frac{3a^2}{4}\right)$

$I_{zz} = 2ma^2 + 3ma^2 = 5ma^2$

$$\underline{\underline{I}} = ma^2 \begin{pmatrix} \frac{9}{4} & -\frac{3\sqrt{3}}{4} & 0 \\ -\frac{3\sqrt{3}}{4} & \frac{11}{4} & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

To find the direction of principal axis we will calculates eigen vector.

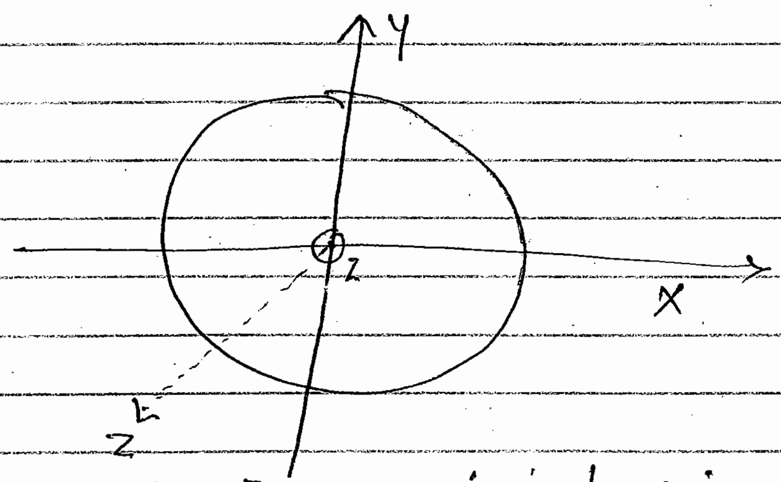
* Principal Axes :- {Defined through a point of rigid body}:-

Physical Defination :-

A symmetric axis is always a principal axis if the object has axial symmetry.

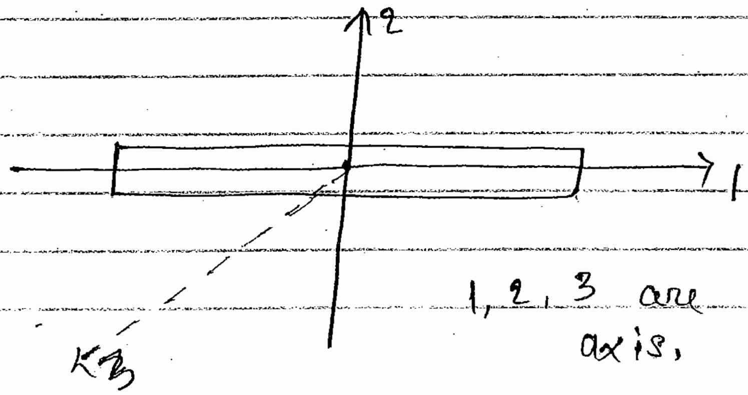
Ex -

① Sphere.



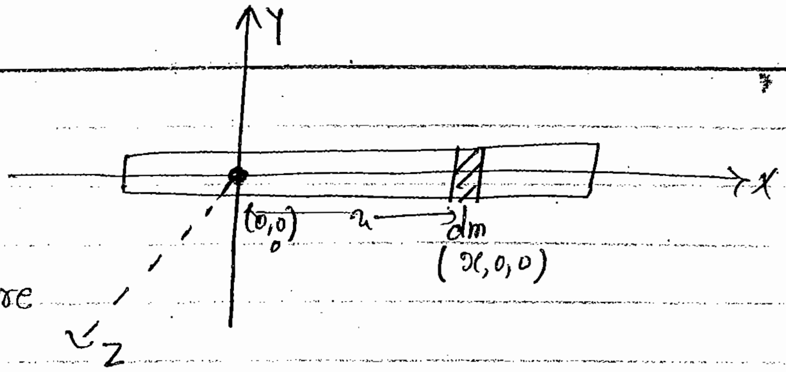
x, y, z are principal axis.

(2) Rod



1, 2, 3 are principal axis.

$\therefore I_{ij} = 0 \quad i \neq j$



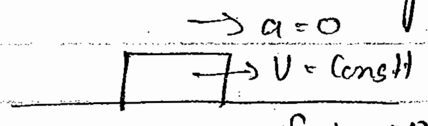
Product of inertia are zero $\therefore$ x, y, z are principal axes.

* Principal axes may or may not be symmetric axes.
Every symmetric axes is a principal axes but every principal axes will not be symmetric.

* Actual Physical Defination of Principal Axes:-

"The set of ~~axex~~ axes about which uniform rotational motion can be maintained without application of a torque are called Principal Axes."

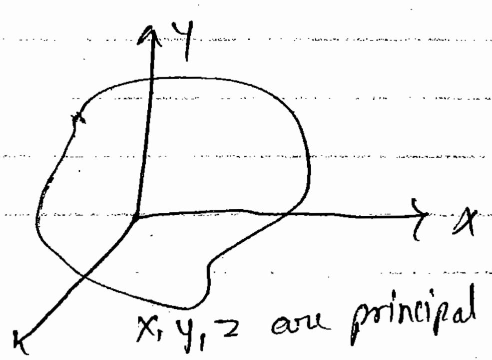
* The axes for which $\vec{L} \parallel \vec{\omega}$ are principal axes.



* Property of principal moment of inertia:-

x, y, z are principal axes
Then,

$$I_{xx} = \sum m_i (y_i^2 + z_i^2)$$
$$I_{yy} = \sum m_i (x_i^2 + z_i^2)$$
$$I_{zz} = \sum m_i (x_i^2 + y_i^2)$$



x, y, z are principal Axes.

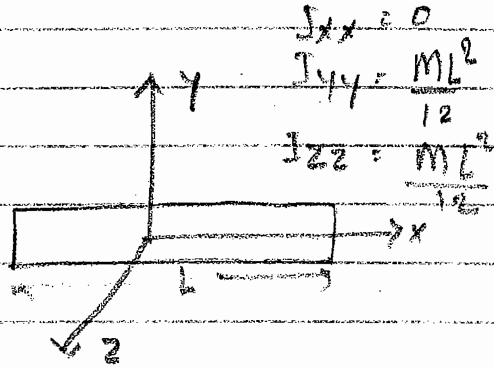
$$I_{xx} + I_{yy} = \sum m_i (x_i^2 + y_i^2 + 2z_i^2)$$

$$I_{xx} + I_{yy} = I_{zz} + 2\sum m_i z_i^2$$

$$\boxed{I_{xx} + I_{yy} - I_{zz} = 2\sum m_i z_i^2 > 0}$$

$$I_{xx} + I_{yy} - I_{zz} \geq 0$$

$$\boxed{I_{xx} + I_{yy} \geq I_{zz}}$$



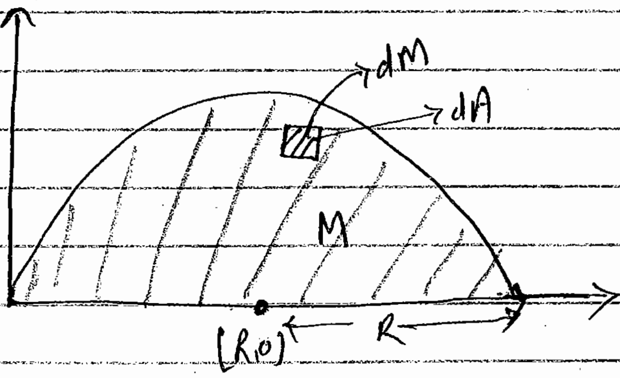
Similarly -

$$\boxed{I_{yy} + I_{zz} \geq I_{xx}} \quad \text{and} \quad \boxed{I_{xx} + I_{zz} \geq I_{yy}}$$

A-6

Q.11 A semicircular disc of Radius R and mass M is kept in x-y plane as shown in figure. The product of inertia I_{xy} is ?

Sol^n



$$I_{xy} = -\int dm\, x \cdot y$$

$$dm = \frac{M}{A} dA = \frac{M}{\frac{\pi R^2}{2}}$$

$$\frac{dm}{dA} = \frac{2M}{\pi R^2}$$

$$dm = \frac{2M}{\pi R^2} dx\,dy$$

$$I_{xy} = -\frac{2M}{\pi R^2} \int\int x\,y\,dx\,dy$$

∵ Here if we moving along the boundary then x and y both change. So limits are dependents.

So first we write equation of boundary –

$$(x-R)^2 + (y-0)^2 = R^2$$

$$\boxed{y = \sqrt{R^2-(x-R)^2}}$$

So $$I_{xy} = -\frac{2M}{\pi R^2}\int_0^{2R} x\,dx \int_0^{\sqrt{R^2-(x-R)^2}} y\,dy$$

$$\cancel{= -\frac{2M}{\pi R^2}\left[\frac{x^2}{2}\right]_0^{2R}\left[\frac{y^2}{2}\right]_0^{\sqrt{R^2-(x-R)^2}}}$$

$$\cancel{= -\frac{2M}{\pi R^2}\left[\frac{4R^2}{2}\right]\left[\frac{R^2-(x-R)^2}{2}\right]}$$

$$\cancel{= -\frac{2M}{4\pi}\left[4\left(R^2-x^2-R^2+2xR\right)\right]}$$

$$= -\frac{2M}{\pi R^2}\int_0^{2R}\left[\frac{y^2}{2}\right]_0^{\sqrt{R^2-(x-R)^2}} x\,dx$$

$$= -\frac{2M}{\pi R^2}\int_0^{2R}\left\{\frac{R^2-(x-R)^2}{2}\right\}\cdot x\,dx$$

$$= -\frac{2M}{\pi R^2}\int_0^{2R}\left[\frac{\cancel{R^2}-x^2-\cancel{R^2}+2xR}{2}\right]x\,dx$$

$$= -\frac{M}{\pi R^2}\int_0^{2R}\left(-x^3+2x^2R\right)dx$$

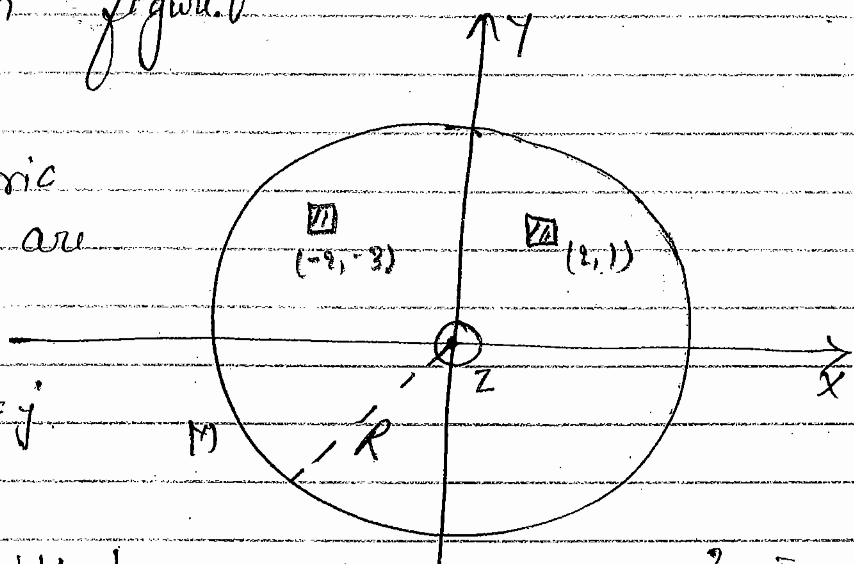
$$= \frac{-M}{\pi R^2}\left[-\left(\frac{x^4}{4}\right)_0^{2R} + 2R\left(\frac{x^3}{3}\right)_0^{2R}\right]$$

$$= \frac{-M}{\pi R^2}\left[\frac{-16R^4}{4} + \frac{2R\times 8R^3}{3}\right] = \frac{-M}{\pi R^2}\left[\frac{-16R^4}{4} + \frac{16R^4}{3}\right]$$

$$= +\frac{16R^4 M}{\pi R^2}\left[\frac{1}{4} - \frac{1}{3}\right] = \frac{{}^{4}\cancel{16}MR^2}{\pi}\left[\frac{3-4}{\cancel{12}_{3}}\right]$$

$$I = -\frac{4MR^2}{3\pi} \quad \underline{\text{Ans}}$$

Q. Write inertia tensor of the disc about the axes shown in figure.



X, Y, Z are symmetric axes so these are principal axes.

$\therefore \; I_{ij} = 0$ for $i \neq j$.

$I_{xx} = \frac{MR^2}{4} = I_{yy}$

$I_{zz} = \frac{MR^2}{2}$

Inertia tensor will be written –

$$\mathbb{I} = \frac{MR^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

* Calculation of I from $\mathbb{I}$ (matrix) :–

B.A.
Q.10

Solⁿ

$$\mathbb{I} = \begin{pmatrix} 8 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 8 \end{pmatrix}$$

Magnitude of M.I about an axis.

$$\hat{n} = \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0\right)$$

$$I = \underset{\substack{\text{row matrix} \\ 1\times 3}}{\hat{n}} \cdot \underset{\substack{3\times 3 \\ \text{Matrix}}}{\overleftrightarrow{\mathbb{I}}} \cdot \underset{\substack{\text{Coloumn matrix} \\ 3\times 1}}{\hat{n}}$$

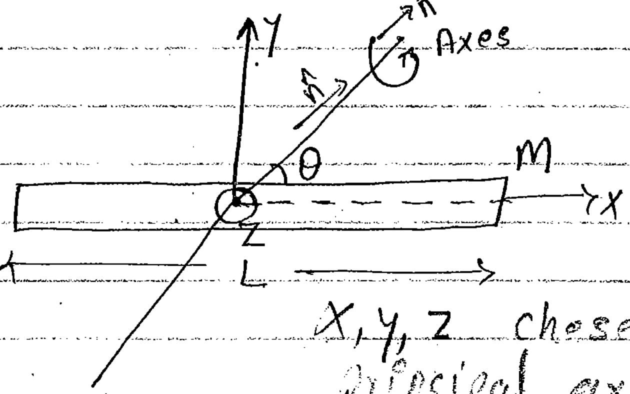
$$I = \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0\right) \begin{pmatrix} 8 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 8 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}$$

$$I = \left(4,\ 2\sqrt{3},\ -2\right) \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \\ 0 \end{pmatrix}$$

$$= 2 + 3 = 5$$

$\boxed{I = 5}$ Ans

Q. Find M.I. of thin rod about the axis shown in fig.



X, Y, Z chosen as principal axis.

$$\mathbb{I} = \frac{ML^2}{12} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{n} = (\cos\theta, \sin\theta, 0)$$

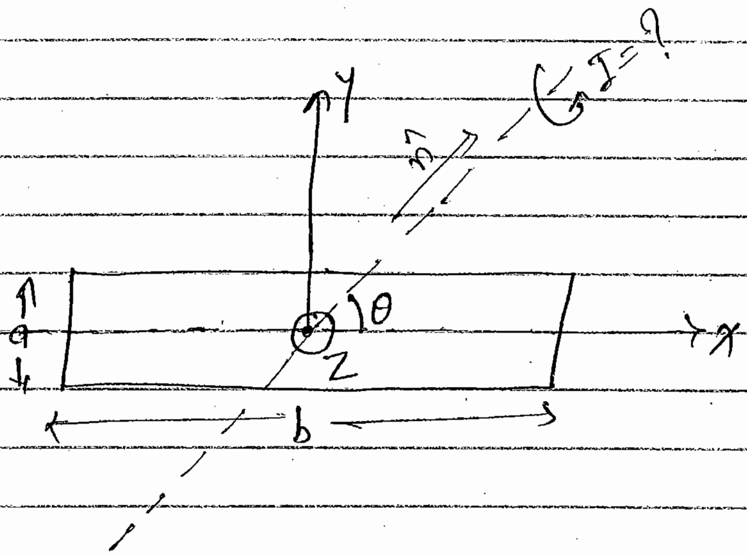
$$I = \hat{n} \cdot \overleftrightarrow{\mathbb{I}} \cdot \hat{n}$$

$$I = (\cos\theta, \sin\theta, 0) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \frac{ML^2}{12}$$

$$= (0, \sin\theta, 0) \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \frac{ML^2}{12}$$

$$\boxed{I = \frac{ML^2}{12} \sin^2\theta}$$ Ans

A-6
Q.21



$$I_{xx} = \frac{Ma^2}{12}$$

$$I_{yy} = \frac{Mb^2}{12}$$

$$I_{zz} = I_{xx} + I_{yy}$$

$$I_{zz} = \frac{M}{12}(a^2+b^2)$$

$$\mathbb{I} = \frac{M}{12} \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & (a^2+b^2) \end{pmatrix}$$

$$\hat{n} = (\cos\theta, \sin\theta, 0)$$

So $I = \hat{n} \cdot \overleftrightarrow{I} \cdot \hat{n}$

$$= \left(\cos\theta, \sin\theta, 0\right) \frac{M}{12} \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & (a^2+b^2) \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

$$= \left(a^2\cos\theta, b^2\sin\theta, 0\right) \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \frac{M}{12}$$

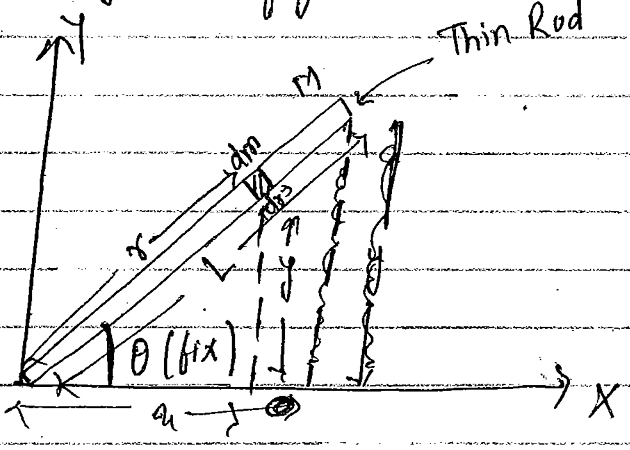
$$= \left(a^2\cos^2\theta + b^2\sin^2\theta\right) \frac{M}{12}$$

$$\boxed{I = \frac{Ma^2}{12}\cos^2\theta + \frac{Mb^2}{12}\sin^2\theta}$$

So option (d) is correct.

Q. Calculate $I_{xy} = ?$ of the given figure.

Sol^n



$$I = -\int dm\, xy$$

$$dm = \frac{M}{L} dr$$

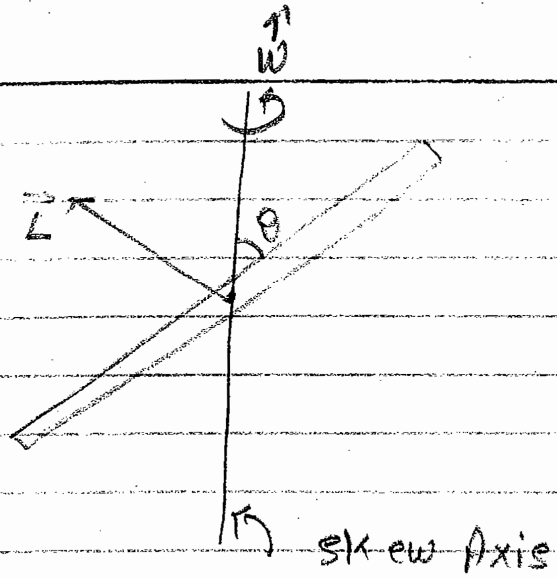
$x = r\cos\theta$
$y = r\sin\theta$

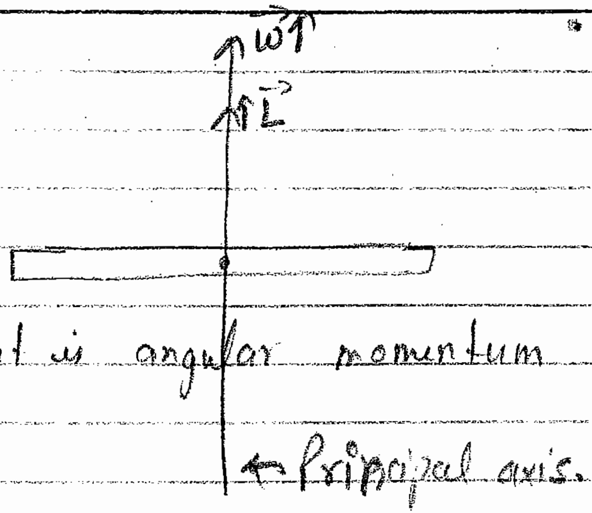
So $I = -\int \frac{M}{L} dr \cdot r\cos\theta \cdot r\sin\theta$

$$= -\frac{M}{L}\cos\theta\sin\theta \int_0^L r^2 dr = -\frac{M}{L}\cos\theta\sin\theta \left[\frac{r^3}{3}\right]_0^L$$

$$\boxed{I = -\frac{ML^2}{3}\sin\theta\cos\theta}$$ Ans

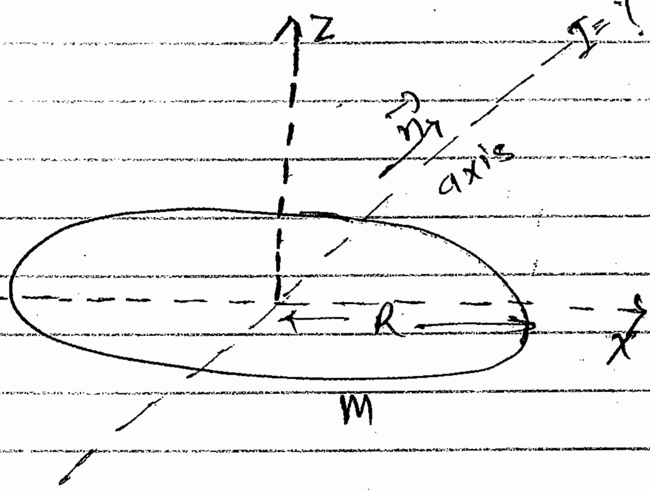
Q.





Q.

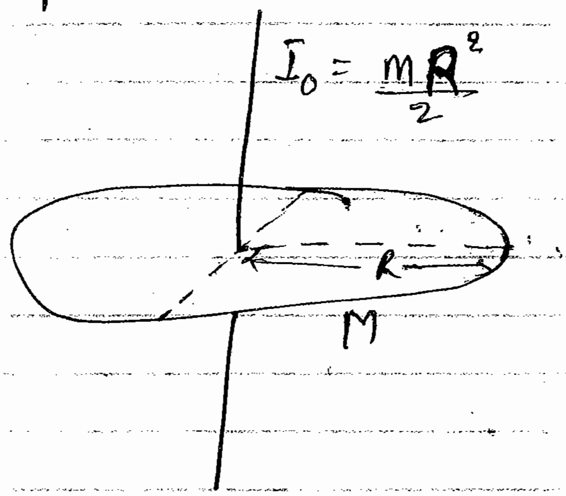
$$\mathbb{I} = \begin{pmatrix} \frac{MR^2}{4} & 0 & 0 \\ 0 & \frac{MR^2}{4} & 0 \\ 0 & 0 & \frac{MR^2}{2} \end{pmatrix}$$

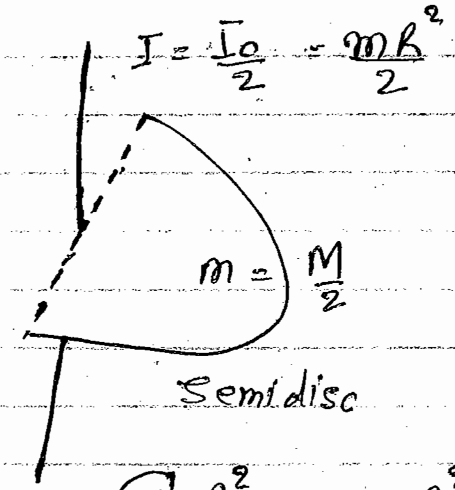


$$\hat{n} = \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$$

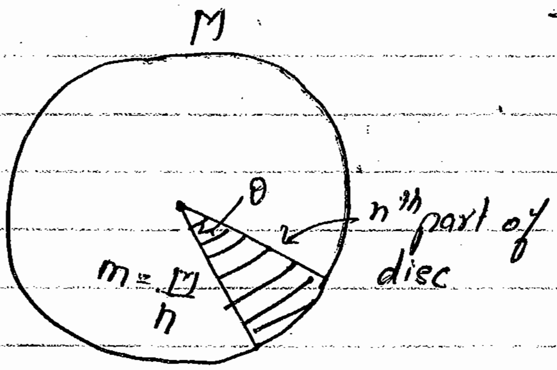
Note: Two important points related to M.I.

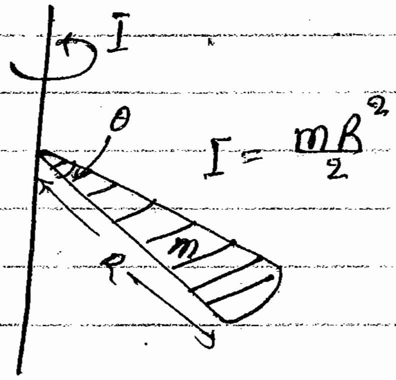
(i) If an object is cut symmetrically then form of M.I. of divided part is same as form of M.I of whole object when expressed in terms of mass of divided part.



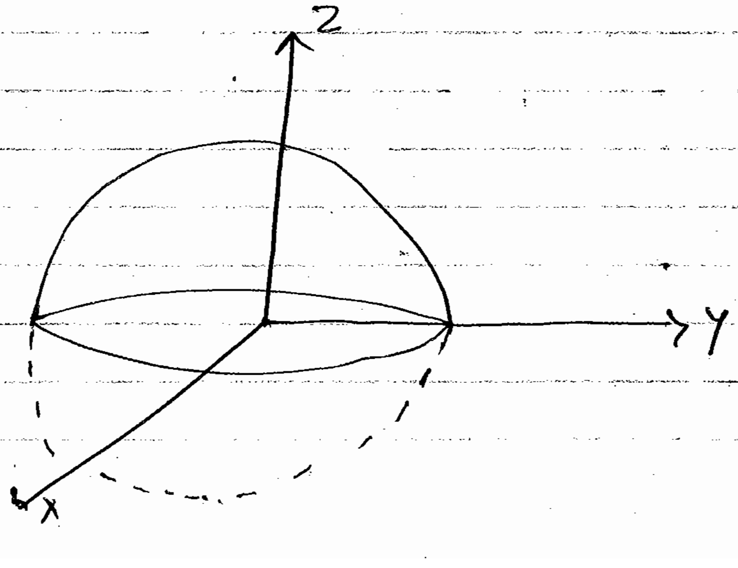


$$I = \frac{I_0}{2} = \frac{M}{2\times 2}R^2 = \frac{mR^2}{2}$$





A-6 Q10

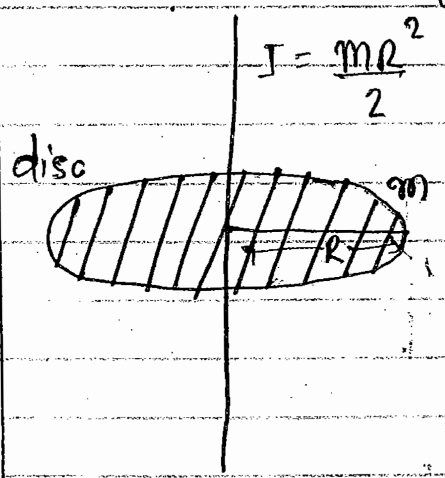


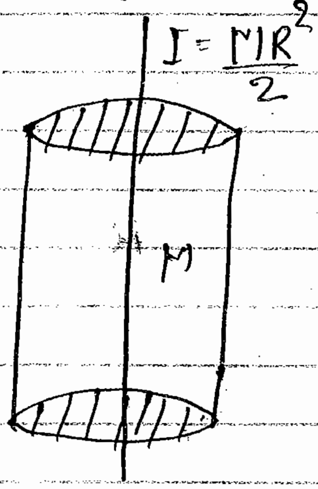
Sol: I about $x = \frac{2}{3}MR^2$

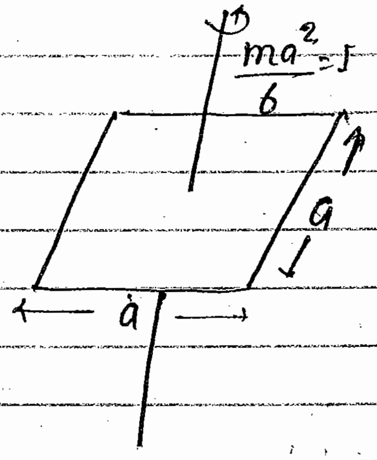
I about $z = \frac{2}{3}MR^2$

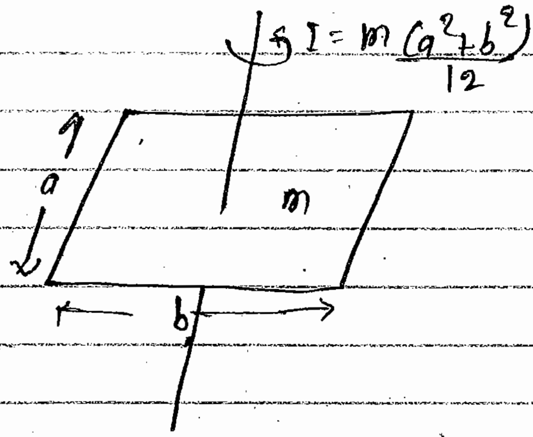
Assembling

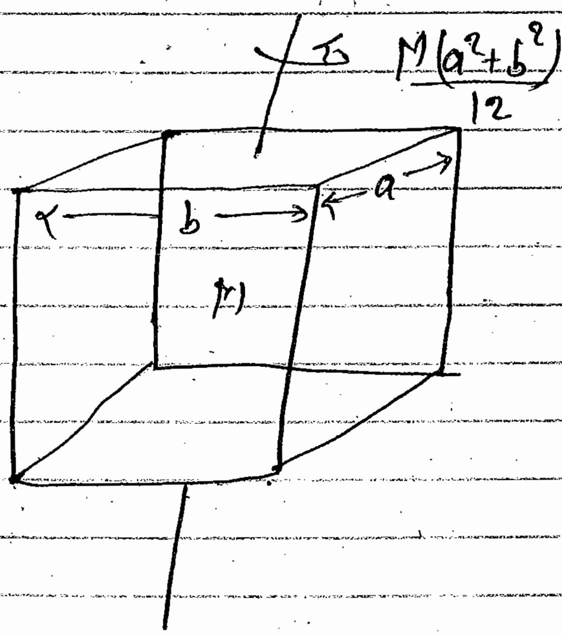
(ii) If stacking of small ~~objejts~~ ~~is~~ objects is done to make a big object then form of M.I. of big objects is same as of M.I. of small objects.

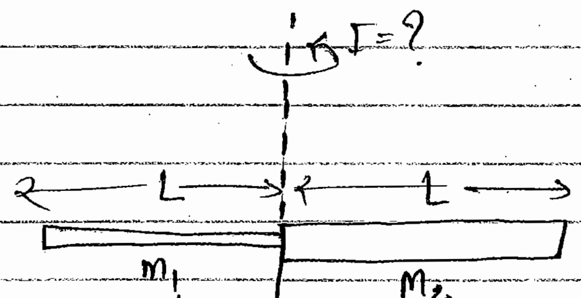












Two thin Rods

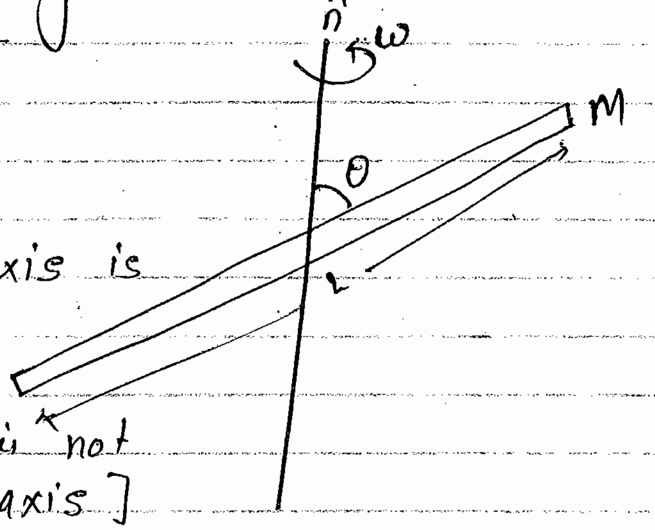
$$I = \frac{m_1 L^2}{3} + \frac{M_2 L^2}{3}$$

$$I = \frac{(m_1 + m_2) L^2}{3}$$

* ## Angular Momentum about skew axis :-

Here two questions can be asked.

* What is the angular momentum about axis.
* What is total angular momentum.



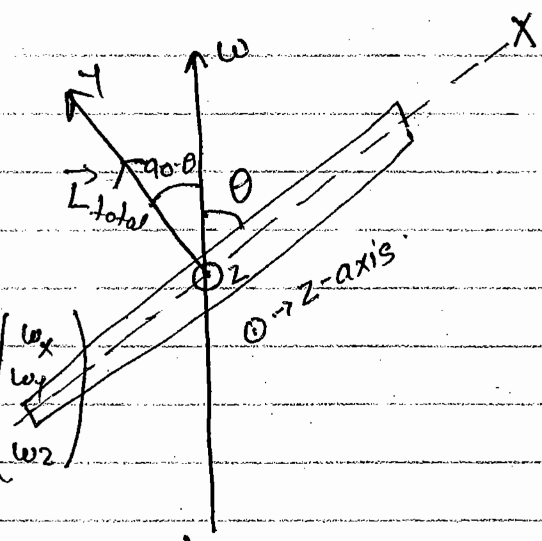
(i) $\boxed{\vec{L}_{axis} = I_{axis}\,\omega}$

(ii) $\vec{L}_{total} = \vec{L}_{axis}$ (if axis is principal axis)

$\vec{L}_{total} = \overleftrightarrow{I}\cdot\vec{\omega}$ [if axis is not a principal axis]

(i) $\vec{L}_{axis} = I_{axis}\cdot\omega$

$$\boxed{\vec{L}_{axis} = \frac{ML^2}{12}\sin^2\theta\cdot\omega}$$



(ii) $\vec{L}_{total} = \overleftrightarrow{I}\cdot\vec{\omega}$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{ML^2}{12} & 0 \\ 0 & 0 & \frac{ML^2}{12} \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{ML^2}{12} & 0 \\ 0 & 0 & \frac{ML^2}{12} \end{pmatrix}\begin{pmatrix} \omega\cos\theta \\ \omega\sin\theta \\ 0 \end{pmatrix}$$

Compairing cofficient:-

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{mL^2}{12}\omega \sin\theta \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} L_x = 0 \\ L_y = \frac{mL^2}{12}\omega\sin\theta \\ L_z = 0 \end{array}$$

$$\vec{L}_{total} = \sqrt{L_x^2 + L_y^2 + L_z^2}$$

$$= \sqrt{0 + \left(\frac{mL^2}{12}\sin\theta\right)^2 + 0}$$

$$\boxed{\vec{L}_{total} = \frac{mL^2}{12}\omega\sin\theta}$$

Angular momentum about axis is just a component of total angular momentum.

$$L_{axis} = L_{y_{total}} \cos(90-\theta)$$

$$= L_{y_{total}} \sin\theta = \frac{mL^2}{12}\omega \sin\theta \cdot \sin\theta$$

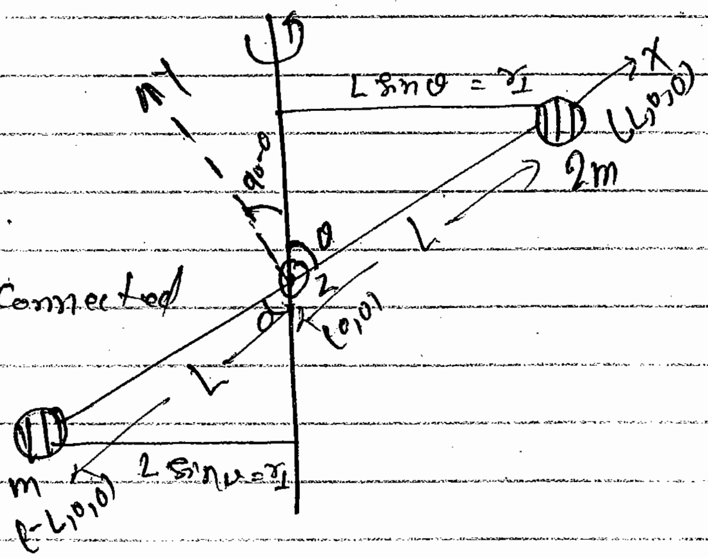
$$\boxed{L_{axis} = L_{total} \frac{mL^2}{12}\omega \sin^2\theta}$$

Ans

Q. find $L_{axis} = ?$

$L_{total} = ?$

Two small particle connected by a small rod.



$$\vec{L}_{axis} = I_{axis} \cdot \omega$$

$$= \sum_{i=1}^{2} \left(m_i r_{\perp i}^2 \right) \omega$$

$$= \left(m \times L^2 \sin^2\theta + 2m \times L^2 \sin^2\theta \right) \omega$$

$$\boxed{\vec{L}_{axis} = 3mL^2 \sin^2\theta \cdot \omega}$$

Total Angular momentum :-

$$\text{Inertia Tensor } \mathbb{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3mL^2 & 0 \\ 0 & 0 & 3mL^2 \end{pmatrix}$$

$I_{yy} = \sum m_i (x_i^2 + z_i^2)$
$= m(l^2 + 0) + 2m \times l^2$
$= 3ml^2$
$I_{zz} = \sum m_i (x_i^2 + y_i^2)$

$$\vec{\omega} = \begin{pmatrix} \omega \cos\theta \\ \omega \sin\theta \\ 0 \end{pmatrix}$$

$$\vec{L}_{total} = \mathbb{I} \cdot \omega = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3mL^2 & 0 \\ 0 & 0 & 3mL^2 \end{pmatrix} \begin{pmatrix} \omega \cos\theta \\ \omega \sin\theta \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 3mL^2 \sin\theta\, \omega \\ 0 \end{pmatrix}$$

$$\boxed{\vec{L}_{total} = L_{2m} + L_m = 3mL^2 \omega \sin\theta.}$$

$$K.E. = \frac{1}{2} I_{axis} \omega^2 = \frac{1}{2} 3 mL^2 \omega \sin^2\theta \cdot \omega^2$$

$$\boxed{K.E. = \frac{3}{2} mL^2 \omega^3 \sin\theta}$$

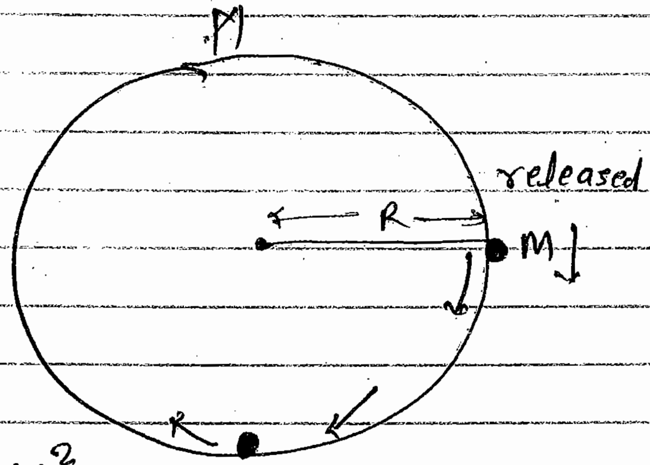
A-7

Q.20 A disc of mass M and Radius R is pivoted about a horizontal axis through its centre and a small body of the same mass M is attached to the rim of the disc. If the disc is released from rest with the small body at the end of a horizontal radius, the angular speed when the small body is at the bottom is?

(a) $\sqrt{\left(\frac{g}{4R}\right)}$ (b) $\sqrt{\left(\frac{g}{2R}\right)}$ (c) $\sqrt{\left(\frac{3g}{4R}\right)}$ (d) $\sqrt{\left(\frac{4g}{3R}\right)}$

Sol^n Now apply Conservation of energy -

Loss in K.E. = Gain in K.E.



$$MgR = \frac{1}{2} I\omega^2$$

$$= \frac{1}{2}\left(\frac{MR^2}{2} + MR^2\right)\omega^2$$

$$\cancel{M}g\cancel{R} = \frac{3\cancel{M}R^{\cancel{2}}}{4}\cdot\omega^2$$

$$\boxed{\omega = \sqrt{\frac{4g}{3R}}}$$

A-7

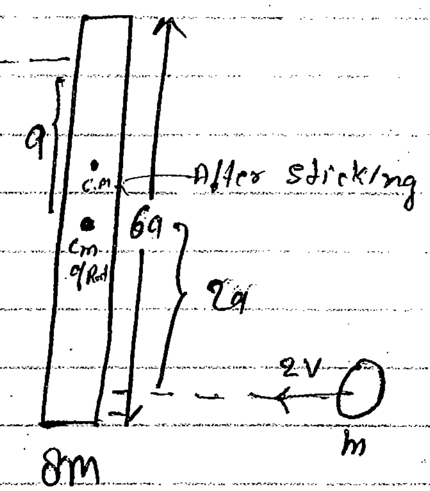
Q.21 An uniform bar of length 6a and mass 8m lies on a smooth horizontal table. Two point masses m and 2m moving in the same horizontal plane with speed 2v and v respectively strike the bar (see fig) and stick to the bar after collision.

Denoting angular velocity (about the center of mass), total energy and centre of mass velocity by ω, E and v_c respectively, we have after collision.

(a) $v_c = 0$ (b) $\omega = \frac{3}{5}\left(\frac{v}{a}\right)$ (c) $\omega = \frac{v}{5a}$ (d) $E = \frac{3}{5} m v^2$

Sol^n The whole system lying on the smooth table therefore $\vec{F}_{ext} = 0$

$$\because \vec{F}_{ext} = \frac{d\vec{p}}{dt}$$
$$\Downarrow$$
$$0 \Rightarrow \vec{p} = \text{Const.}$$



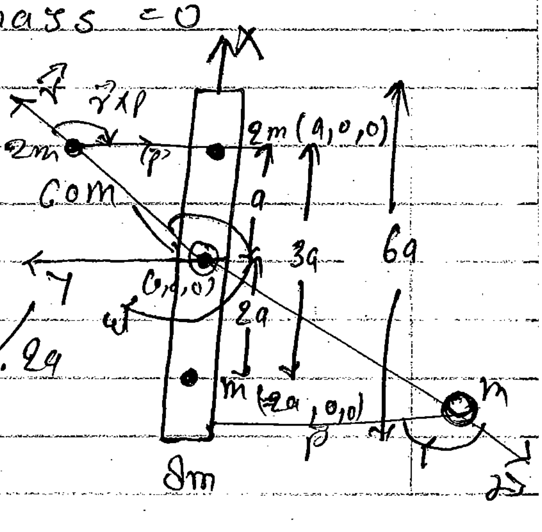
Now apply Conservation of momentum

$$P_i = P_f$$

$$\cancel{2mv - m2v} = P_f$$

$$\boxed{P_f = 0} \Rightarrow \boxed{V_c = 0}$$

So Here initial and final momentum is $= 0$
So velocity of centre of mass $= 0$



$$X_{cm} = \frac{\sum m_i x_i}{\sum m_i}$$

$$= \frac{2m \cdot a + 8m \cdot 0 - m \cdot 2a}{11a}$$

$$= 0$$

So C.M. of system is mid of the rod.

Here $F_{ext} = 0$ therefore there is no external force So there is no external torque.

$$\vec{\tau} = \frac{d\vec{L}}{dt} = 0$$

So $\vec{L}$ = Constant

So we apply Conservation of Angular momentum

$$L_i = L_f$$

$$v \times 2m \times a + m \times 2v \times 2a = I\omega$$

$$6mav = I\omega$$

$$6mva = \left(\underset{\substack{\uparrow \\ \text{Rod}}}{\frac{MR^2}{12}} + \underset{\substack{\uparrow \\ \text{Particle}}}{m_1 r_1^2} + \underset{\substack{\uparrow \\ \text{Particle}}}{m_2 r_2^2}\right)\omega$$

$$= \left[\frac{8m(6a)^2}{12} + 2ma^2 + m(2a)^2\right]\omega$$

$$6mva = \left(\frac{8m \cdot \cancel{36}^{3} a^2}{\cancel{12}} + 2ma^2 + 4a^2m\right)\omega$$

$$\cancel{6}\cancel{m}v\cancel{a} = \frac{30}{5}\cancel{m}a^2\omega$$

$$\boxed{\omega = \frac{v}{5a}}$$

Total Energy : Pure Rotation:-

$$\text{Energy} = \frac{1}{2} I\omega^2$$

$$= \left(\frac{1}{\cancel{2}} \cancel{30}^{\cancel{6}\,3} m\cancel{a^2} \times \frac{v^2}{\cancel{25}_{5}\cancel{a^2}}\right)$$

$$\boxed{E = \frac{3mv^2}{5}}$$

Ans

11|Sep|2014

Special theory of Relativity

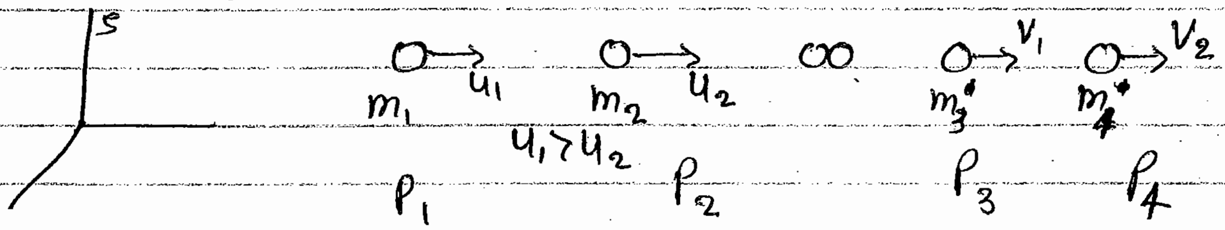
Relativistic Mechanics (It deals with dynamics at very high speed $v \sim c$) :-

It will reduce in to newtonian mechanics if $v \ll c$ or $c \longrightarrow \infty$.

- Special theory of Relativity deals with Inertial frame of reference.

* Postulates of STR :-

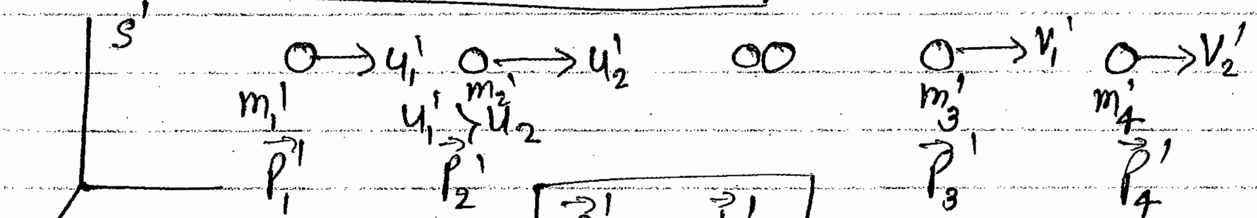
(i) Laws of physics are invariant in all inertial frame of reference. It means that mathematical form of the law remains unchanged in all inertial frame.



Law of conservation of momentum -

$$\vec{p_i} = \vec{p_f}$$

$$\boxed{\vec{P_1} + \vec{P_2} = \vec{P_3} + \vec{P_4}}$$



$$\boxed{\vec{P_i}' = \vec{P_f}'}$$

$$\vec{P_1}' + \vec{P_2} \neq \vec{P_3}' + \vec{P_4}'$$

In classical mechanics $c \to \infty$ where c is speed of information. In classical mechanics we tells that information is travelling in infinite speed.

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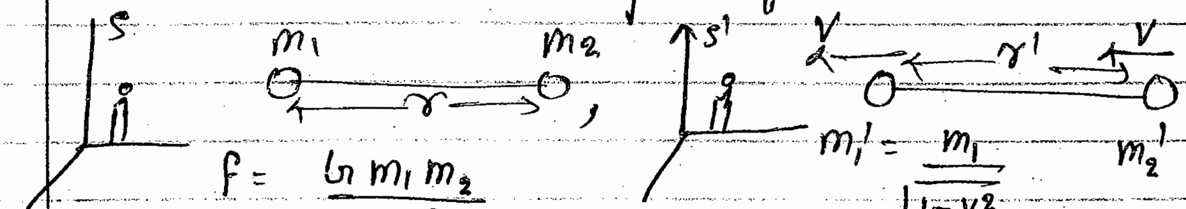
* If mathematical form of the law doesn't remain invariant then that law should be modified.
for example:- Newton's Law of gravitation changes its form in different frames due to which Einstein modified it in his General theory of relativity.



$$F = \frac{G m_1 m_2}{r^2}$$

$$m_1' = \frac{m_1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad m_2' = \frac{m_2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$r' = r\sqrt{1-v^2/c^2}$$

$$\vec{F}' = \frac{G m_1' m_2'}{r'^2}$$

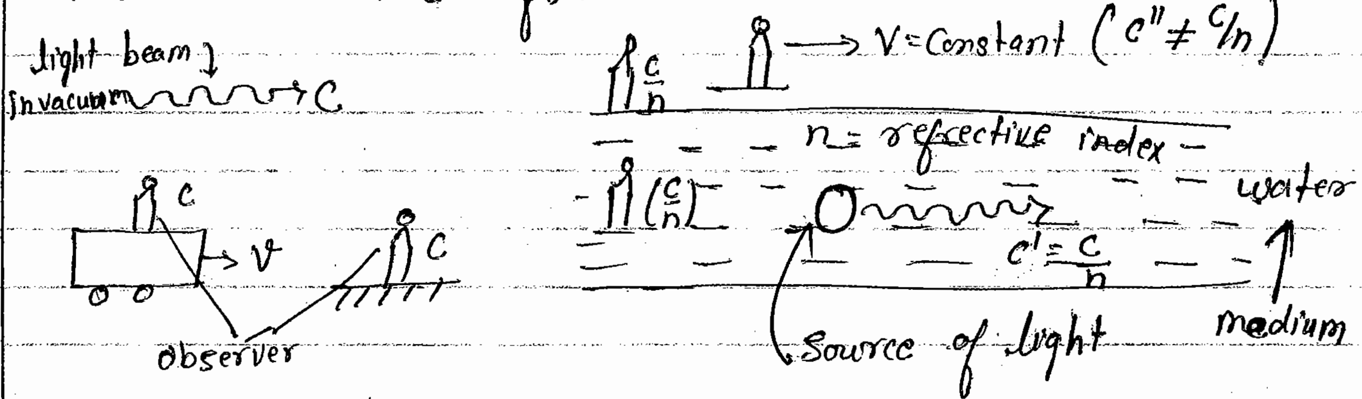
$$\boxed{\vec{F}' = \frac{G m_1 m_2}{r^2\left(1-\frac{v^2}{c^2}\right)^2}}$$

So newton's law of gravitation is not true.

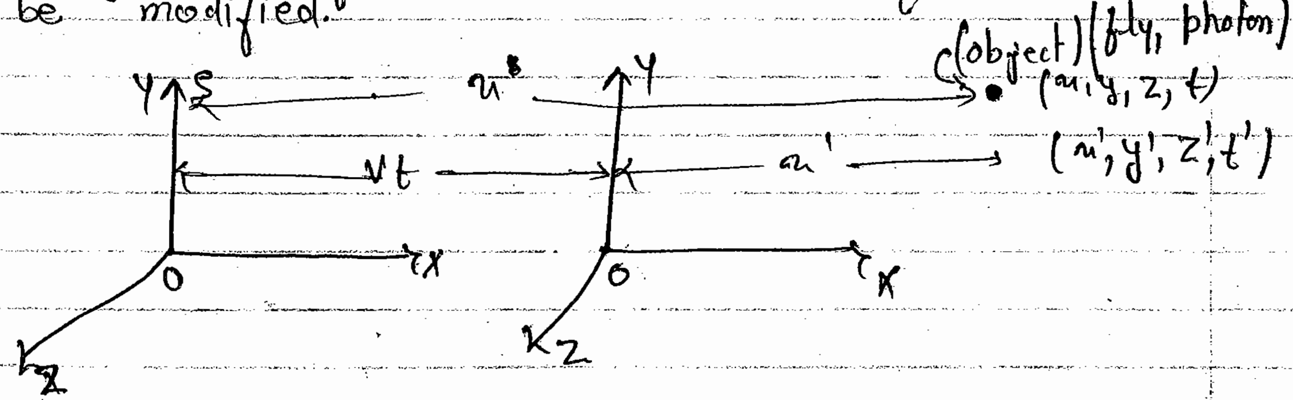
⇒

$$\nabla^2 E = \frac{1}{\mu_0 \varepsilon_0} \frac{\partial^2 E}{dt^2} = \frac{1}{c^2} \frac{d^2 E}{dt^2}$$

* **Second Postulate of Relativity :-** It states that speed of light ~~in vacc~~ (magnitude of resultant velocity) in vacuum is same in an inertial frame and it is also ultimate speed.
* Direction may change.
* Components may chance $\{ c = \sqrt{c_x^2 + c_y^2 + c_z^2} \}$.
* If there is a medium involved then speed of light will not be same in different ~~invert~~ inertial frame.



* **Lorentz's Transformation:-** Einstein said due to relative motion geometric relation must change therefore Galilean transformation must be modified.



Note :- Einstein using his two postulatee show that $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

See :- Aurthur Beiser's book.

Galilian Transformation :-

$$\left.\begin{aligned} x &= x' + vt \\ x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\}$$ G. T. for Coordinate
It's a Geometric relation
or Euclidean Geometry

$$\left.\begin{aligned} v'_x &= v_x - v \\ v_y' &= v_y \\ v_z' &= v_z \end{aligned}\right\}$$ G-T for velocity

if light is viewed from S and S' its speed will be different in S and S'
So Galilian transformation is must be modified.

Here in $x' = x - vt$ for modification we can't change its linearty. becoz if we change their linearty then some contradiction will arises. Like if we modified x' in place of x by x^2 then we find two value. So at a same time we find two position. So modification is takes in that manner :-

L. T. :- {Direct Transformation}

$$\left.\begin{aligned} x' &= (x - vt)\gamma \\ y' &= y \\ z' &= z \\ t' &= \left(t - \frac{xv}{c^2}\right)\gamma \end{aligned}\right\}$$ It is valid if S' is moving in x direction.

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Einstein using his two postulates showed that

$$\boxed{\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}}$$

* L.T tells us that measurement has been done in s frame and we are concluding what the results will be in s'.

* Inverse L-T :-

$$x = (x' + vt)\gamma$$
$$y = y'$$
$$z = z'$$
$$t = \left(t' + \frac{x'v}{c^2}\right)\gamma$$

Here $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$$s' \longrightarrow s$$

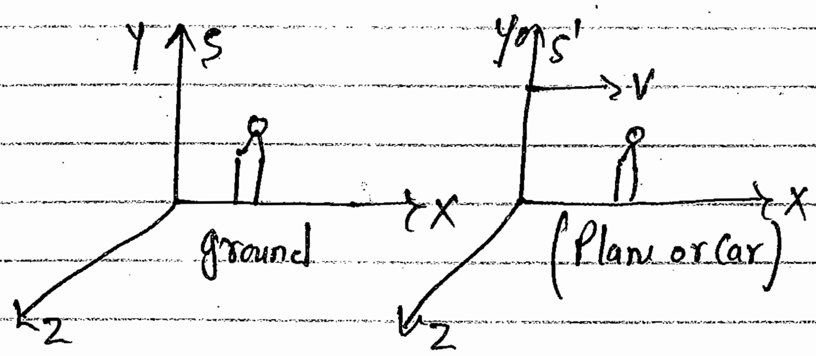
* Differential form of L-T :-

Direct
$$\begin{cases} \Delta x' = (\Delta x - v\Delta t)\gamma \\ \Delta t' = \left(\Delta t - \frac{v\Delta x}{c^2}\right)\gamma \end{cases}$$

$$\begin{cases} \Delta x = (\Delta x' + v\Delta t')\gamma \\ \Delta t = \left(\Delta t' + v\frac{\Delta x'}{c^2}\right. \end{cases}$$

These are used when time difference of coordinate are to be calculated.

* Velocity Transformation {Addition}:-



Object

(x, y, z) (V_x, V_y, V_z) in s'

(x', y', z') (V_x', V_y', V_z') in s'

$$V_x = \frac{dx}{dt}$$

$$V_x' = \frac{dx'}{dt'} = \frac{(dx - v\,dt)\cancel{\gamma}}{\left(dt - \frac{v\,dx}{c^2}\right)\cancel{\gamma}}$$

$$= \frac{\left(\frac{dx}{dt} - v\right)}{\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{V_x - V}{1 - \frac{V_x V}{c^2}}$$

$$\boxed{V_x' = \frac{V_x - V}{1 - \frac{V}{c^2} V_x}} \qquad S \longrightarrow S'$$

$$V_y = \frac{dy}{dt}$$

$$V_y' = \frac{dy'}{dt'} =$$

So

$$V_x' = \frac{V_x - V}{\left(1 - \frac{V_x V}{c^2}\right)}$$

$$V_y' = \frac{V_y \sqrt{1 - V^2/c^2}}{\left(1 - \frac{V_x . V}{c^2}\right)}$$

$$V_z' = \frac{V_z \sqrt{1 - V^2/c^2}}{\left(1 - \frac{V_x V}{c^2}\right)}$$

mesurement in $S \longrightarrow S'$

Inverse :-

$$V_x = \frac{V_x' + V}{1 + \frac{V_x V}{c^2}}$$

$$V_y = \frac{V_y' \sqrt{1 - V^2/c^2}}{1 + V_x' V/c^2}$$

$$V_z = \frac{V_z' \sqrt{1 - V^2/c^2}}{1 + \frac{V_x' V}{c^2}}$$

mesurement is done in $S' \longrightarrow S$.

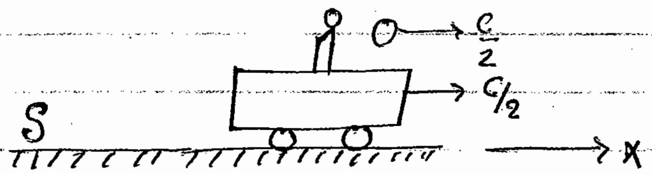
Q. A car is moving horizontally with speed $\frac{c}{2}$ a ball is thrown from the car with speed $c/2$ in the direction of movement of the car. What is velocity of ball as seeing from ground.?

Sol^n Given - $V = c/2$

$V_x' = c/2$

$V_y' = 0$

$V_z' = 0$



So $$V_x = \frac{V_x' + V}{1 + \frac{V_x' V}{c^2}}$$

$$= \frac{\frac{c}{2} + \frac{c}{2}}{1 + \frac{1}{4}} = \frac{4c}{5}$$

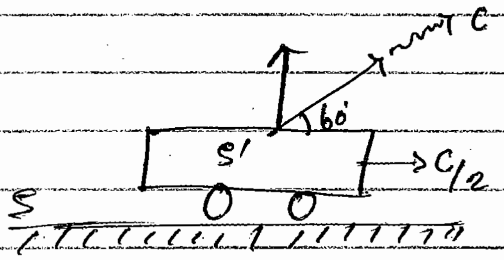
$$V_y = \frac{V_y' \sqrt{1 - V^2/c^2}}{1 + \frac{V_x' V}{c^2}} = 0$$

$$V_z = \frac{V_z' \sqrt{1 - V^2/c^2}}{1 + \frac{V_x' V}{c^2}} = 0$$

Q. A car is moving with speed $c/2$ a light beam is emitted from the car at angle $60°$ with the direction of velocity of car. What is component of velocity of light in the direcn of velocity of car as seen from ground.

Soln

given $V = c/2$



$$V_x' = c\cos 60 = \frac{c}{2}$$

$$V_y' = c\sin 60 = \frac{\sqrt{3}c}{2}$$

$$V_x = \frac{V_x' + V}{1 + \frac{V_x' V}{c^2}} = \frac{\frac{c}{2} + \frac{c}{2}}{1 + 1/4}$$

$$= \frac{4c}{5} \quad \underline{\text{Ans}}$$

Extending :-

$$V_y = \frac{V_y' \sqrt{1 - v^2/c^2}}{1 + \frac{V_x' v}{c^2}}$$

$$= \frac{\frac{c\sqrt{3}}{2}\sqrt{1-\frac{1}{4}}}{1+\frac{1}{4}} = \frac{3c}{5}$$

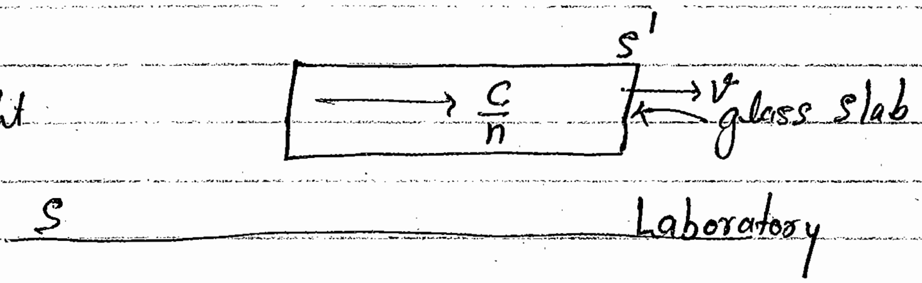
Resultant :- in S :-

$$\sqrt{V_x^2+V_y^2} = \sqrt{\left(\frac{4}{5}c\right)^2+\left(\frac{3c}{5}\right)^2} = c = \text{velocity of light,}$$

A-8

Q52 A beam of light moves in a slab of glass of refractive index n in the positive x-direction. The slab itself is also moving in the positive dire^n with a speed v in laboratory frame. What is the speed of the beam of light as measured in the laboratory frame?

Sol^n



Velocity of light w.r.to glass

$= \frac{c}{n}$

$V_x' = \frac{c}{n}$

$$V_x = ? = \frac{V_x'+v}{1+\frac{V_x' \cdot v}{c^2}} = \frac{\frac{c}{n}+v}{1+\frac{\frac{c}{n}\cdot v}{c^2}}$$

$=$

* If there are two objects then the object w.r.to. which speed is to be calculate is taken as S' frame.

Gate-2012

Q. An e^- and photon is moving in same direcn in lab frame. if speed of e^- is 0.85c what is speed of e^- w.r.to. photon?

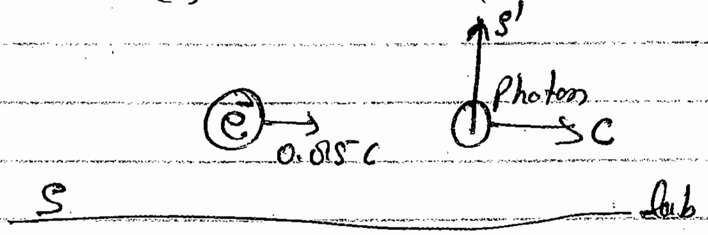
(a) c (b) −c (c) 0.15c (d) −0.15c

Soln

$V_x = 0.85c$

$V_x' = ?$

$V = c$



$$V_x' = \frac{V_x - V}{1 - \frac{V_x \cdot V}{c^2}} = \frac{0.85c - c}{1 - \frac{0.85c \cdot c}{c^2}}$$

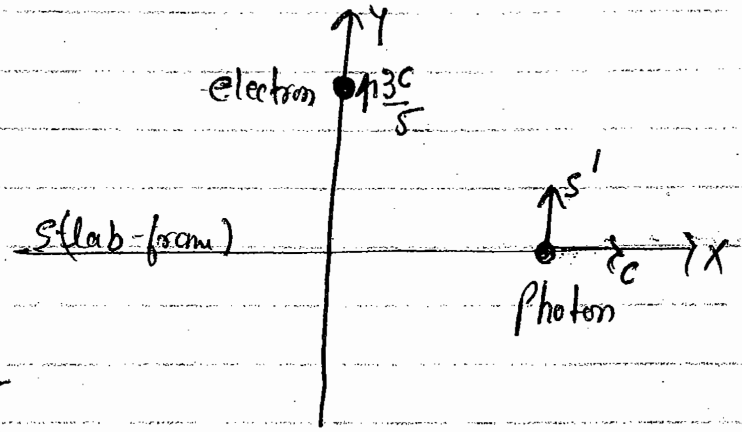
$$= \frac{-0.15c}{0.15} = -c \quad \text{Ans}$$

{ when e^- is goin in any angle then ans. same.

* Velocity of any object w.r.to photon is $\pm c$ (means magnitude is always c)
If photon is going in positive direction then it is $-c$ and when photon is going in negative direction is then it is $+c$.

Q.13

Sol^n



$V = c$

$V_x = 0$

$V_y = \frac{3c}{5}$

$$V_x' = \frac{V_x - V}{1 - \frac{V_x \cdot V}{c^2}}$$

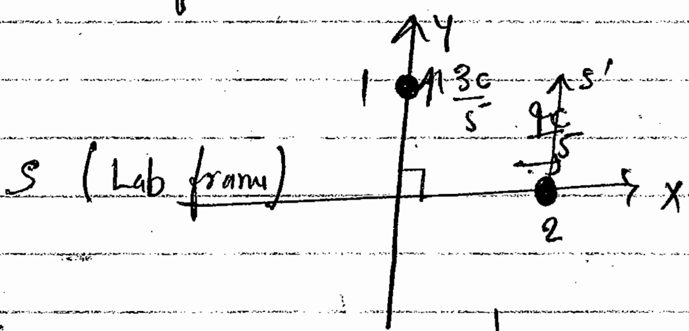
$$= \frac{0 - c}{1 - 0} = -c$$

$$V_y' = \frac{V_y\sqrt{1 - V^2/c^2}}{1 - \frac{V_x \cdot V}{c^2}} = \frac{\frac{3c}{5}\sqrt{1-1}}{1-0} = 0$$

So option (d) is correct.

Q.12 Two electrons are moving as shown in the figure. Velocity of 1 with respect to 2 is?

Sol^n



$V = \frac{4c}{5}$

$V_x = 0$

$V_y = \frac{3c}{5}$

So $V_x' = \frac{V_x - V}{1 - \frac{V_x \cdot V}{c^2}}$

$$= \frac{0 - \frac{4c}{5}}{1 - 0} = \frac{-4c}{5}$$

$$V_y' = \frac{V_y\sqrt{1 - V^2/c^2}}{1 - \frac{V_x \cdot V}{c^2}}$$

$$= \frac{\frac{3c}{5}\sqrt{1-\frac{16}{25}}}{1-0} = \frac{9}{25}c$$

Resultant velocity of 1 w.r. to 2:-

$$= \sqrt{v_x'^2 + v_y'^2} = \sqrt{\left(\frac{-4c}{5}\right)^2 + \left(\frac{9c}{25}\right)^2}$$

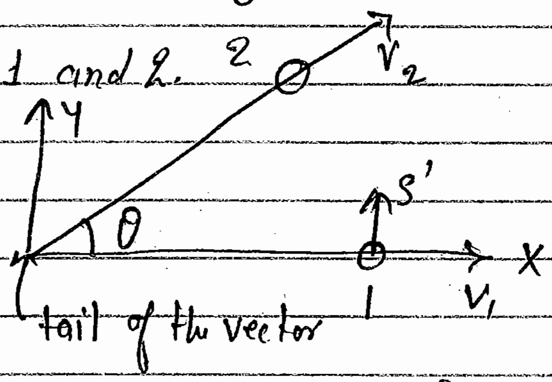
$$= \sqrt{\frac{16c^2}{25} + \frac{81}{625}c^2}$$

$$= \frac{\sqrt{481}}{25}c$$

Ans

* General Formula for relative velocity of two object moving at same angle:-

Relative velocity b/w 1 and 2.



$$|v_{12}| = |v_{21}|$$

$$\vec{v}_{12} = -\vec{v}_{21}$$

* Non-Relativistic Case: $\{v_1, v_2 \ll c\}$

$$\boxed{|v_{12}| = |\vec{v}_{21}| = \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos\theta}}$$

* Relativistic Case:-

Let us calculate -

v_{21} (Velocity of 2 w.r.t. 1).

So $\quad v = v_1$

$$v_x = V_2 \cos\theta$$
$$v_y = V_2 \sin\theta$$

So $$v_x' = \frac{v_x - V}{1 - \frac{v_x . V}{c^2}} = \frac{V_2 \cos\theta - V_1}{1 - \frac{V_2 \cos\theta . V_1}{c^2}}$$

$$v_y' = \frac{v_y \sqrt{1 - V^2/c^2}}{1 - \frac{v_x . V}{c^2}} = \frac{V_2 \sin\theta \sqrt{1 - \frac{V_1^2}{c^2}}}{1 - \frac{V_1 V_2 \cos\theta}{c^2}}$$

Resultant velocity (in s' frame):-

i.e. Resultant velocity of 2 w.r. to 1.

$$|V_{21}| = \sqrt{V_x'^2 + V_y'^2}$$

$$= \frac{\sqrt{(V_2 \cos\theta - V_1)^2 + V_2^2 \sin^2\theta \left(1 - \frac{V_1^2}{c^2}\right)}}{1 - V_1 V_2 \cos\theta / c^2}$$

$$\boxed{|V_{21}| = \frac{\sqrt{V_1^2 + V_2^2 + 2 V_1 V_2 \cos\theta - \frac{V_1^2 V_2^2 \sin^2\theta}{c^2}}}{1 - \frac{V_1 . V_2}{c^2} \cos\theta}}$$

If $V_1, V_2 << C$

$$V_1^2 . V_2^2 \sim 0$$

$$\frac{V_1 V_2}{c^2} \sim 0$$

Q.12

Sol^n $V_1 = \frac{3c}{5}$, $V_2 = \frac{4c}{5}$, $\theta = 90°$

$$|V_{21}| = |V_{12}| = \sqrt{\left(\frac{3c}{5}\right)^2 + \left(\frac{4c}{5}\right)^2 - \frac{\left(\frac{3c}{5}\right)^2\left(\frac{4c}{5}\right)^2}{c^2}}$$

$$= \sqrt{\frac{9c^2}{25} + \frac{16c^2}{25} - \frac{\frac{9c^2}{25} \times \frac{16c^2}{25}}{c^2}}$$

$$= \sqrt{\left(\frac{\cancel{25}}{\cancel{25}}\right)c^2 - \frac{144c^{\cancel{4}}}{625\cancel{c^2}}}$$

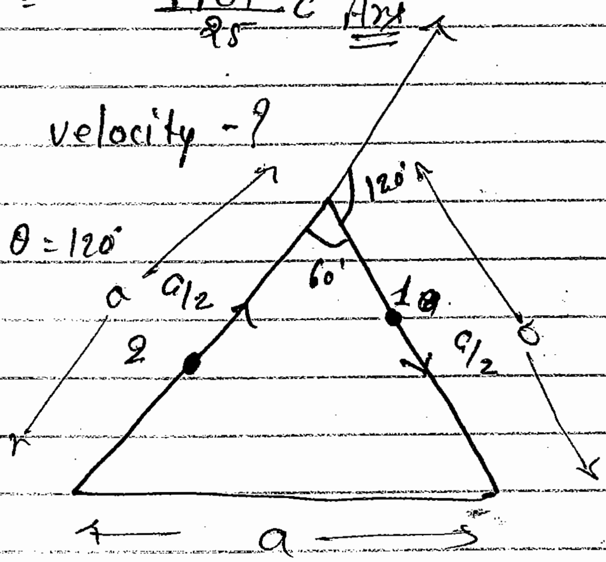
$$= \sqrt{c^2 - \frac{144c^2}{625}} = \sqrt{\left(1 - \frac{144}{625}\right)c^2}$$

$$= c\cancel{\sqrt{\frac{89}{125}}} = c\sqrt{\frac{625-144}{625}}$$

$$= c\sqrt{481/625} = \frac{\sqrt{481}}{25}c \text{ Ans}$$

2012
TIFR
Q. What is their relative velocity -?

Sol^n $V_1 = c/2$, $V_2 = c/2$, $\theta = 120°$

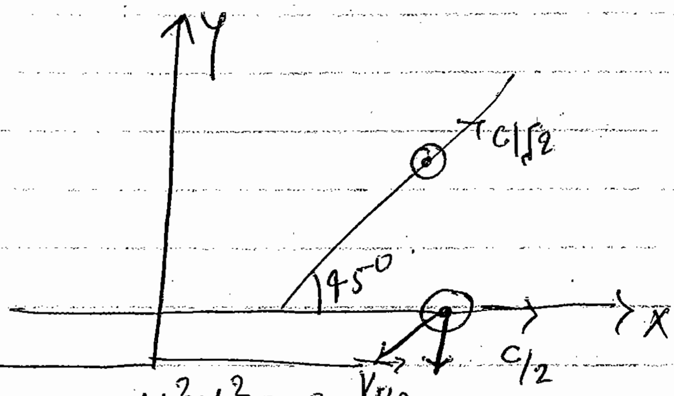


A-8

Q.50

$V_{AB} = ?$

$\vec{V}_{BA} = -\vec{V}_{AB}$



$$|V_{12}| = \frac{\sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos\theta - \frac{v_1^2 v_2^2 \sin^2\theta}{c^2}}}{1 - \frac{v_1 v_2 \cos\theta}{c^2}}$$

$$= \frac{\sqrt{\frac{c^2}{2} + \frac{c^2}{4} - 2\frac{c}{\sqrt{2}}\frac{c}{2}\frac{1}{\sqrt{2}} - \frac{\frac{c^2}{2}\frac{c^2}{4}\frac{1}{2}}{c^2}}}{1 - \frac{2\frac{c}{\sqrt{2}}\frac{c}{2}\cdot\frac{1}{\sqrt{2}}}{c^2}}$$

$$= \frac{\sqrt{\frac{c^2}{4} - \frac{c^2}{16}}}{1 - \frac{1}{4}} = \frac{c\sqrt{\frac{1}{4} - \frac{1}{16}}}{(1 - \frac{1}{4})}$$

$$= \frac{\frac{c}{2}\sqrt{1 - \frac{1}{4}}}{3/4} = \frac{c\sqrt{1 - \frac{1}{4}}}{3/2} \quad \frac{\sqrt{3}c}{2} \times \frac{2}{3}$$

$$\boxed{|V_{12}| = \frac{c}{\sqrt{3}}}$$ Ans

Second Method :-

Let us calculate -

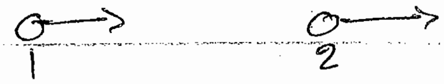
$$V_{u}' = \frac{V_u - V}{1 - \frac{V_u V}{c^2}} = \frac{\frac{c}{2} - \frac{c}{2}}{1 - \frac{c}{4}} = 0$$

$$V_y' = \frac{V_y \sqrt{1 - V^2/c^2}}{1 - \frac{V_x V}{c^2}} = \frac{\frac{c}{2}\sqrt{1 - \frac{1}{4}}}{1 - \frac{1}{4}} = \frac{c}{\sqrt{3}}$$

$$\boxed{V_y' = \frac{c}{\sqrt{3}}}$$

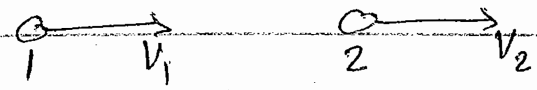
Non Relativistic Case :-

$$|V_{12}| = \sqrt{V_1^2 + V_2^2 - 2V_1V_2\cos\theta}$$



$$\boxed{V_{12} = V_1 - V_2}$$

Relativistic Case :-



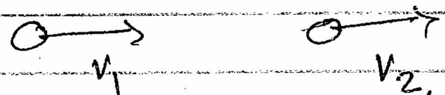
$$V_{12} = V_x' = \frac{V_x - V}{1 - \frac{V_x \cdot V}{c^2}}$$

$$\boxed{V_{12} = \frac{V_1 - V_2}{1 - \frac{V_1 V_2}{c^2}}}$$

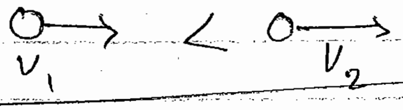
* Velocity of approach / Separation :-

$$= |\vec{V_1} - \vec{V_2}|$$

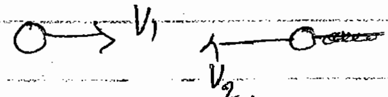
In both relativistic and non relativistic case.



Velocity of approach $= V_1 - V_2$



$$\boxed{\text{Velocity of separation} = v_2 - v_1}$$



$$\text{Velocity of approach} = v_1 - (-v_2)$$
$$= v_1 + v_2$$

* When two moving objects are observed by a stationary observer then velocity of approach is used for time calculation.

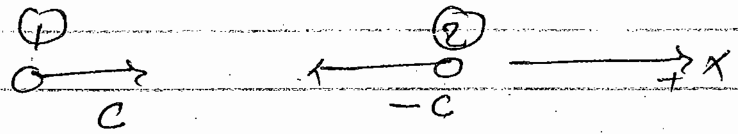
* If one moving object is observed by another moving object then relative velocity is used for time calculation.

* Velocity of approach may be less equal or greater than velocity of light.

* Relative velocity is never be greater than velocity of light.

A-8
Qu#7

Q.8 In the previous question velocity of approach of the two photons is -



Velocity of approach $= |V_1 - V_2|$

$= |V_1 - (-V_2)|$

$= |c - (-c)|$

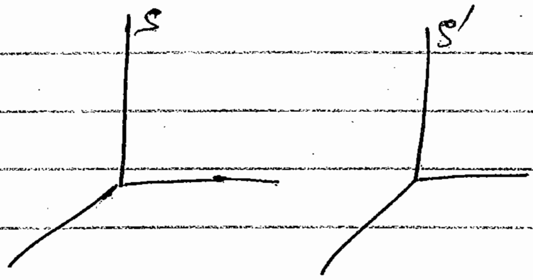
$= 2c$ Ans

Q.62 Velocity of a particle is $\vec{V} = \frac{c}{\sqrt{2}}(\hat{i} + \hat{j})$ in s frame and $\vec{V'} = \frac{c}{\sqrt{2}}(-\hat{i} + \hat{j})$ in s' frame which is moving along x direction w.r.to S velocity of s' w.r.to. S frame is ?

Sol^n

$\vec{V} = \frac{c}{\sqrt{2}}(\hat{i} + \hat{j})$ in s

$\vec{V}' = \frac{c}{\sqrt{2}}(-\hat{i} + \hat{j})$ in s'



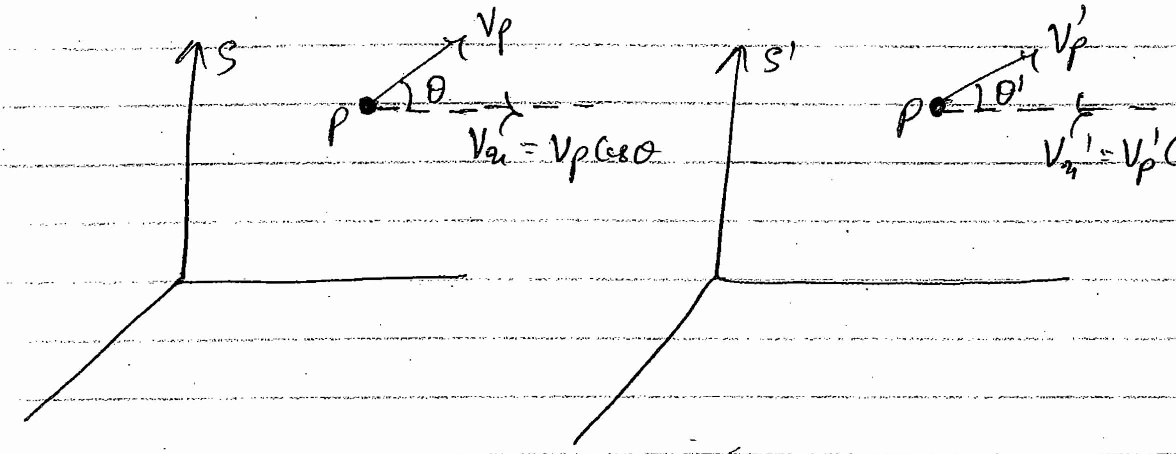
$$V_x' = \frac{V_x - V}{1 - \frac{V_x \cdot V}{c^2}}$$

$$\frac{-c}{\sqrt{2}} = \frac{\frac{c}{\sqrt{2}} - V}{1 - \frac{\frac{c}{\sqrt{2}} \cdot V}{c^2}}$$

$$-\frac{c}{\sqrt{2}} + \frac{V}{2} = \frac{c}{\sqrt{2}} - V$$

$$\frac{3V}{2} = \frac{\sqrt{2}\,2c}{\sqrt{2}} \Rightarrow \boxed{V = \frac{2\sqrt{2}c}{3}}$$

* Direction velocity vector in two frames :-



Let v_p and v_p' be speed of a particle as as seen from S and S' frames.
Let θ and θ' be its angle with x-dirⁿ

$$\cos\theta = \frac{v_x}{v_p} \qquad \cos\theta' = \frac{v_x'}{v_p'}$$

$$v_p = \sqrt{v_x^2 + v_y^2 + v_z^2} \;, \quad v_p' = \sqrt{v_x'^2 + v_y'^2 + v_z'^2}$$

{ Calculate component of velocity by lorentz transformation then we get angle. }

or
$$\boxed{\tan\theta = \frac{v_y}{v_x}}$$
$$\boxed{\tan\theta' = \frac{v_y'}{v_x'}}$$

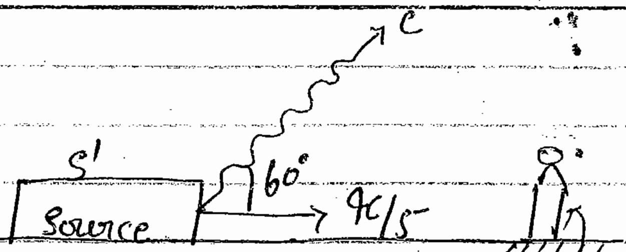
JEST 2013

Q. A monocromatic wave propagate in the direcⁿ making an angle 60° with X-axis in refrence frame of the source. The source moves ~~towards~~ at a speed $\frac{4c}{5}$ towards the observer the direⁿ of wave as seen by observer is

(a) $\cos^{-1}\frac{13}{14}$ (b) $\cos^{-1}\frac{3}{14}$ (c) $\cos^{-1}\frac{\sqrt{3}}{6}$ (d) $\cos^{-1}\left(\frac{1}{2}\right)$.

Monocromatic → Electromagnetic wave.

Sol^n.



$V_x = V\cos 60 = \frac{c}{2}$

Let θ is the angle as seen by ~~obser ver~~ observer.

$$\cos\theta = \frac{V_x}{V_p} = \frac{V_x}{c}$$

$$V_x = \frac{V_x' + V}{1 + \frac{V_x' \cdot V}{c^2}} = \frac{\frac{c}{2} + \frac{9c}{5}}{1 + \frac{\frac{c}{2} \cdot \frac{9c}{5}}{c^2}}$$

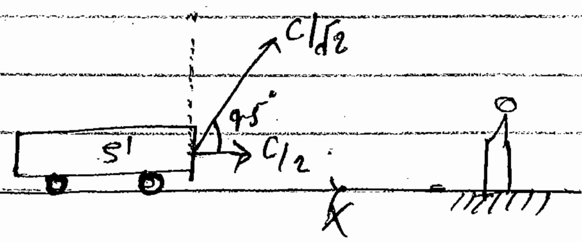
$$\boxed{V_x = \frac{13c}{14}}$$

$$\cos\theta = \frac{V_x}{c} = \frac{13\cancel{c}}{14\cancel{c}} = \frac{13}{14}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{13}{14}\right)}$$ Ans

2. A car is moving with speed $c/2$ a stone is ~~throug~~ thrown from the car with speed $c/\sqrt{2}$ at 45° with the direc^n of velocity of the car. what is the angle as seen from the ground.

Sol^n



$$V_x' = \frac{c}{\sqrt{2}}\cos 45 = \frac{c}{2}$$

$$V_y' = \frac{c}{\sqrt{2}}\cos 45 = \frac{c}{2}$$

$$V_x = \frac{V_x' + V}{1 + \frac{V_x' \cdot V}{c^2}} = \frac{\frac{c}{2} + \frac{c}{2}}{1 + \frac{1}{4}} = \frac{4c}{5}$$

$$V_y' = \frac{V_y'\sqrt{1-V^2/c^2}}{1+\frac{V_x V}{c^2}} = \frac{\frac{c}{2}\sqrt{1-\frac{1}{4}}}{1+\frac{1}{4}} = \frac{\sqrt{3}c}{5}$$

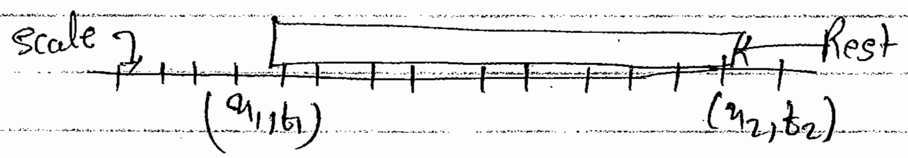
$$V_p = \sqrt{V_x^2+V_y^2} = \frac{\sqrt{19}\,c}{5}$$

$$\boxed{\cos\theta = \frac{V_x}{V_p} = \frac{4}{\sqrt{19}}}$$

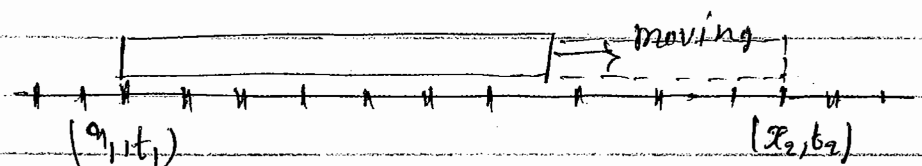
* Relativistic of Geometry :-

* Length Contraction :-

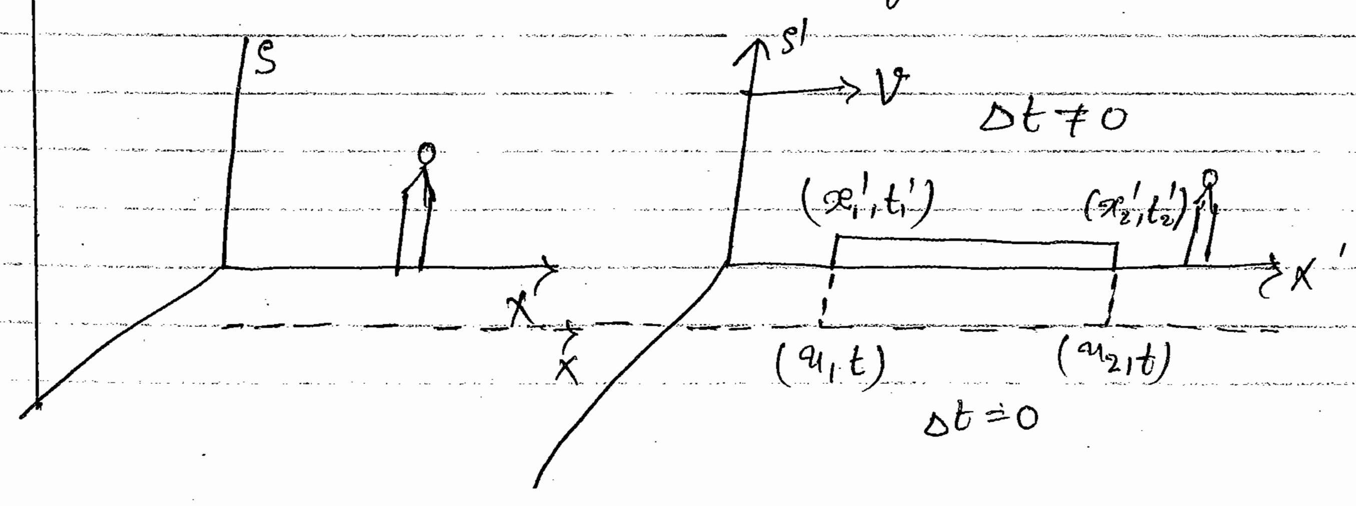
If length of a moving object is to be measured its end cordinates must be noted simultaneously. Then cordinate difference will be equal to length.



$x_2 - x_1 = \text{length}$



$x_2 - x_1 \neq \text{length}.$



V is the velocity parallel to the rod.

Rod is lying in frame S'.

$\Delta x'$ = rest length = L_0 = Proper length
$\Delta x'$ = length in motion or apparent length = L

$$\Delta x' = \frac{\Delta x - V\Delta t}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\Delta x = \frac{\Delta x' + V\Delta t}{\sqrt{1 - V^2/c^2}}$$

$$L_0 = \frac{L - 0}{\sqrt{1 - V^2/c^2}}$$

$$\boxed{L = L_0\sqrt{1 - \frac{V^2}{c^2}}}$$

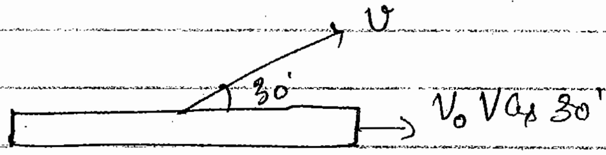
$$\boxed{L = L_0\sqrt{1 - \frac{V_{\parallel}^2}{c^2}}}$$

$V_{\parallel}$ = Relative velocity $\parallel^{\text{lel}}$ to the length of the rod.

A-8
Q.1

Sol^n

$$L = \frac{L_0}{2}$$



$$L = L_0\sqrt{1 - V_{\parallel}^2/c^2}$$

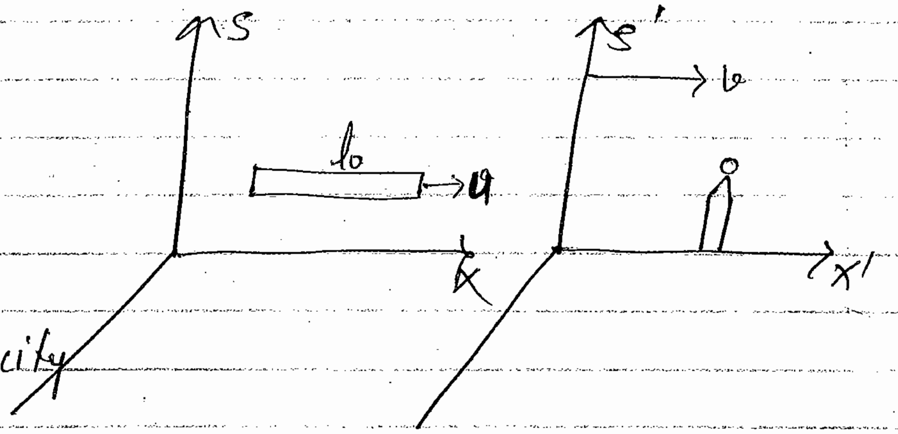
$$\frac{\cancel{L_0}}{2} = \cancel{L_0}\sqrt{1 - V^2 \frac{3}{4c^2}}$$

$$\frac{1}{4} = 1 - \frac{3}{4}\frac{V^2}{c^2}$$

$$\boxed{V = c}$$

Q.19 A rod of proper length l_0 oriented parallel to the x-axis

Sol^n



$$L = L_0\sqrt{1 - \frac{V_{11}^2}{c^2}}$$

V_{11} = relative velocity

Relative Velocity $V_u' = \dfrac{V_u - V}{1 - \frac{V_u V}{c^2}} = \dfrac{u - v}{1 - \frac{uv}{c^2}} = V_{11}$

$$L = L_0\sqrt{1 - \frac{1}{c^2}\left(\frac{(u-v)^2}{\left(1 - \frac{uv}{c^2}\right)^2}\right)}$$

$$= \frac{l_0}{c\left(1 - \frac{uv}{c^2}\right)}\sqrt{c^2\left(1 - \frac{uv}{c^2}\right)^2 - (u-v)^2}$$

$$= \frac{l_0}{c\left(1 - \frac{uv}{c^2}\right)}\sqrt{c^2\left(1 + \frac{u^2v^2}{c^4} - \cancel{\frac{2uv}{c^2}}\right) - \left(u^2 + v^2 - \cancel{2uv}\right)}$$

$$= \frac{l_0}{c\left(1-\frac{uv}{c^2}\right)} \sqrt{c^2 + \frac{u^2v^2}{c^2} - u^2 - v^2}$$

$$= \frac{l_0}{c\left(1-\frac{uv}{c^2}\right)} \sqrt{(c^2-v^2) + u^2\left(\frac{v^2}{c^2}-1\right)}$$

$$= \frac{l_0}{c\left(1-\frac{uv}{c^2}\right)} \sqrt{c^2\left(1-\frac{v^2}{c^2}\right) - u^2\left(1-\frac{v^2}{c^2}\right)}$$

$$= \frac{l_0}{c\left(1-\frac{uv}{c^2}\right)} \sqrt{\left(1-\frac{v^2}{c^2}\right)(c^2-u^2)}$$

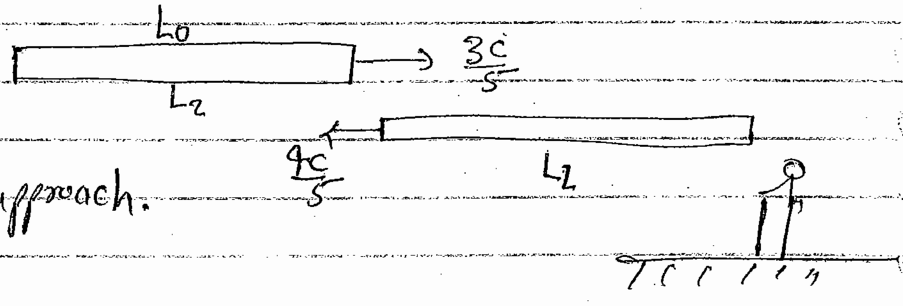
$$= \frac{l_0}{c\left(1-\frac{uv}{c^2}\right)} \sqrt{\left(1-\frac{v^2}{c^2}\right) c^2\left(1-\frac{u^2}{c^2}\right)}$$

$$\boxed{l = \frac{l_0}{\left(1-\frac{uv}{c^2}\right)} \sqrt{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u^2}{c^2}\right)}}$$

Ans

Q.14 Two rods of rest length L_0 are moving towards each other with speeds $\frac{3c}{5}$ and $\frac{4c}{5}$. what is time taken by the two rods to cross each other as seen by a person on the ground.

Sol^n



$$\text{time} = \frac{\text{distance}}{\text{Velocity of approach.}}$$

$$L_1 = L_0\sqrt{1-\frac{v^2}{c^2}} = L_0\sqrt{1-\frac{\left(\frac{3c}{5}\right)^2}{c^2}}$$

$$= L_0\sqrt{1-\frac{9\cancel{c^2}}{25\cancel{c^2}}} = L_0\sqrt{\frac{25-9}{25}}$$

$$\boxed{L_1 = \frac{4L_0}{5}}$$

$$L_2 = L_0\sqrt{1-\frac{V^2}{c^2}} = L_0\sqrt{1-\frac{16\cancel{c^2}}{25\cancel{c^2}}}$$

$$\boxed{L_2 = \frac{3L_0}{5}}$$

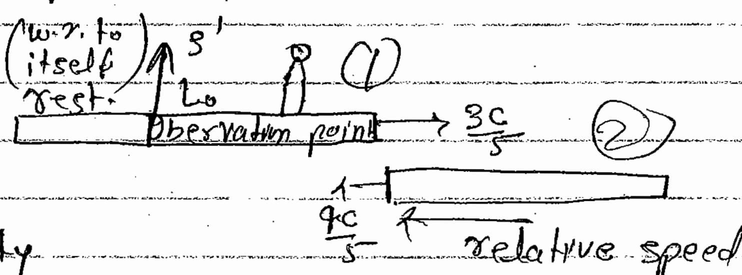
$$t = \frac{\text{distance}}{\text{velocity of approach}} = \frac{L_1+L_2}{(V_1-[-V_2])}$$

$$= \frac{\frac{4L_0}{5}+\frac{3L_0}{5}}{\frac{3c}{5}+\frac{4c}{5}} = \frac{7L_0/5}{7c/5} = \frac{L_0}{c}$$

$$\boxed{t = \frac{L_0}{c}}$$

Q.15 In previous ques what is time taken by the two rods to cross each other as seen by the rod of speed $\frac{3c}{5}$.

Sol^n



$$t = \frac{\text{distance}}{\text{relative velocity}}$$

Relative speed $\quad V_x' = \dfrac{V_x - V}{1-\frac{V_x . V}{c^2}} = \dfrac{-\frac{4c}{5}-\frac{3c}{5}}{1+\frac{4\cancel{c}}{5}\cdot\frac{3\cancel{c}}{5}}$

$$\boxed{V_{4}' = -\frac{35}{37}c}$$

Apparant length of rod (2) :-

$$L_2 = L_0\sqrt{1-\frac{V_{11}^2}{c^2}} = L_0\sqrt{1-\frac{\left(\frac{-35}{37}\right)^2 \cancel{c^2}}{\cancel{c^2}}}$$

$$= L_0\frac{\sqrt{(37)^2-(35)^2}}{37}$$

$$= L_0\frac{\sqrt{(37+35)(37-35)}}{37}$$

$$= L_0\frac{\sqrt{72\times 2}}{37} = \frac{L_0\sqrt{144}}{37}$$

$$\boxed{L_2 = \frac{12L_0}{37}}$$

So $\quad t = \dfrac{L_1+L_2}{\frac{35}{37}c} = \dfrac{L_0+\frac{12L_0}{37}}{\frac{35}{37}c}$

$$= \frac{49L_0/\cancel{37}}{35c/\cancel{37}} = \frac{7L_0}{5c}$$

$$\boxed{t = \frac{7L_0}{5c}}$$ Ans

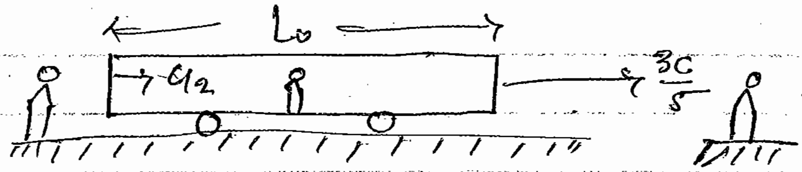
rear end = back end.

Q. A car of rest length L_0 is moving with speed $V = \frac{3c}{5}$. A stone is thrown from rear end of the car with speed $c/2$ relative to the car. What is the time after which the stone will hit front end of the car as measured in car frame and also as measured in ground frame.

Sol

In Car frame :-

$$\text{time} = \frac{\text{dist.}}{\text{relative velo.}}$$
(here speed of stone)



$$= \frac{L_0}{c/2}$$

$$\boxed{t' = \frac{2L_0}{c}}$$

In ground frame :-

$$t = \frac{\text{dis.}}{\text{Velocity of approach}}$$

$$L = L_0\sqrt{1 - V_{11}^2/c^2} \qquad = L_0\sqrt{1 - \frac{9c^2}{25c^2}}$$

$$= L_0\frac{\sqrt{25-9}}{5} \qquad = L_0\frac{\sqrt{16}}{5}$$

$$L = \frac{L_0 4}{5} = \frac{4L_0}{5}$$

$$\Rightarrow \boxed{L = \frac{4L_0}{5}}$$

Velocity of stone w.r. to ground :-

$$= |\vec{V_1} - \vec{V_2}|$$

$$V_{u} = \frac{V_{u}' + V}{1 + \frac{V_u V}{c^2}} = \frac{\frac{c}{2} + \frac{3c}{5}}{1 + \frac{3c^2}{10c^2}}$$

$$\boxed{V_x = \frac{11c}{13}}$$

$$\text{time} = \frac{L_0\,4/5}{\left|\frac{3c}{5} - \frac{11c}{13}\right|} = \frac{4L_0/5}{\frac{(39-55)\,c}{5\times 13}}$$

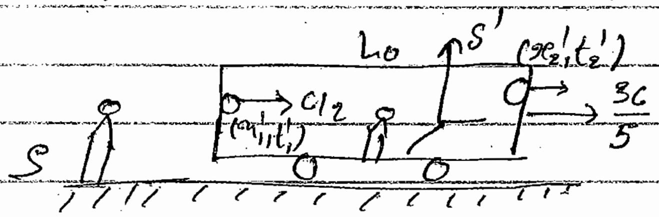
$$= \frac{4L_0 \times 13}{\cancel{16}_4\, c} = \frac{13L_0}{4c}$$

$$\boxed{t = \frac{13L_0}{4c}}$$

Ans

Second Method :- By Lorentz's Transformation

$\Delta x' = x_2' - x_1' = L_0$



$\Delta t' = t_2' - t_1' = \frac{2L_0}{c}$

$\Delta t = ?$

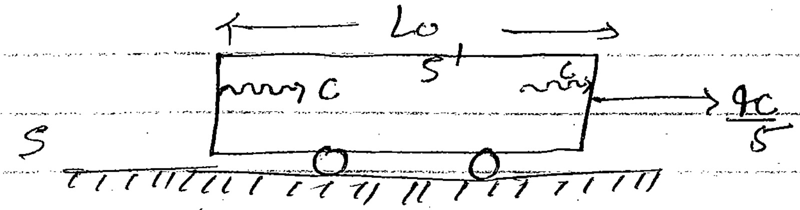
$$\Delta t = \frac{\Delta t' + \Delta x' \frac{v}{c^2}}{\sqrt{1 - v^2/c^2}} = \frac{\frac{2L_0}{c} + L_0\,\frac{3c}{5}/c^2}{\sqrt{1 - \frac{9}{25}}}$$

$$= \frac{\frac{2L_0}{c} + \frac{3L_0}{5c}}{4/5} = \frac{13L_0}{4c}$$

Ans

Q. A car of rest length L_0 is moving with speed $\frac{4c}{5}$ a light beam is emited from rear end of the car and it is absorbed at the front end. What is distance traveled by the car during this process.

Sol^n



In S' frame :-

$$\Delta t' = \frac{L_0}{c}$$

$$\Delta x' = L_0$$

Time duration of process in ground frame :-

$$\Delta t = \frac{\Delta t' + \frac{\Delta x' V}{c^2}}{\sqrt{1 - V^2/c^2}} = \frac{\frac{L_0}{c} + \frac{L_0 \frac{4}{5} c}{c^2}}{3/5}$$

$$\boxed{\Delta t = \frac{3L_0}{c}}$$

Distance travelled by car = speed × time in ground frame.

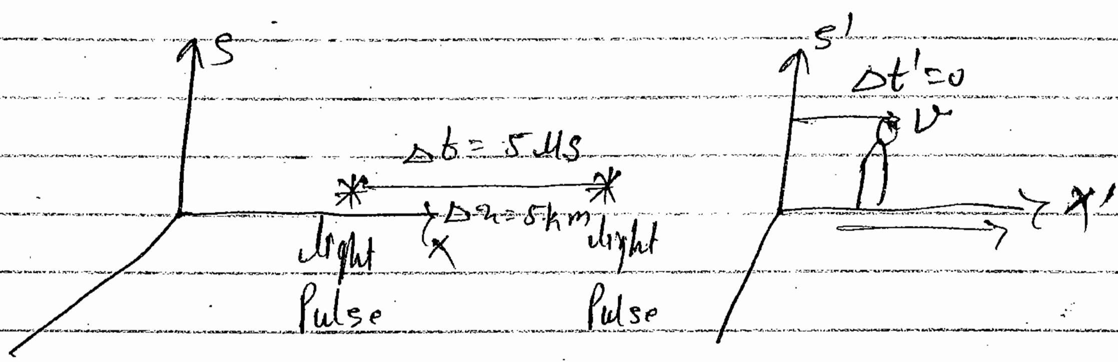
$$= \frac{4\cancel{c}}{5} \times \frac{3L_0}{\cancel{c}}$$

$$= \frac{12 L_0}{5}$$ Ans

JEST 2014

2. In a certain inertial frame two light pulses are emited, a distance ~~distance~~ 5 km apart and separated by 5 μs. An observer who is travelling parallel to the line ~~going~~ joining the two points where pulses are emited at a velocity v w.r.t. this frame note that the pulses are simultaneous. therefore v is?

Sol^n



$$\Delta t' = \frac{\Delta t - \frac{\Delta x v}{c^2}}{\sqrt{1 - v^2/c^2}}$$

$$0 = 5\times 10^{-6} - \frac{5\times 10^{3} v}{c^2}$$

$$\cancel{5}\times 10^{-6} = \frac{\cancel{5}\times 10^{3} v}{c^2}$$

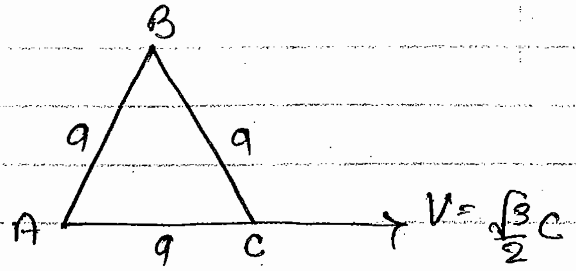
$$v = c^2 \times 10^{-9}$$

$$= c \times 3\times 10^{8} \times 10^{-9}$$

$$\boxed{v = 0.3c}$$ Ans

A-9
Q.2

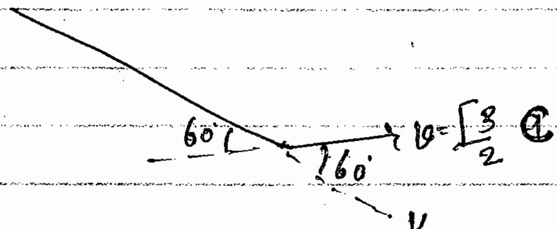
Proper ~~length~~ / Rest perimeter $= 3a$.



AC :- $A \longrightarrow C$, $V = \frac{\sqrt{3}}{2}c$

$$AC = a\sqrt{1-\frac{3}{4}} = \frac{a}{2}$$

BC :-

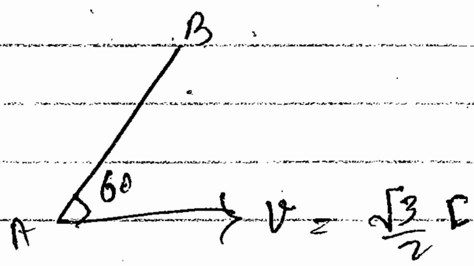


$$V_{\parallel} = V\cos 60 = \frac{\sqrt{3}}{4}c$$

$$BC = a\sqrt{1-\frac{V_{\parallel}^2}{c^2}}$$

$$= a\sqrt{1-\frac{3}{16}} = \frac{a\sqrt{13}}{4}$$

AB :-



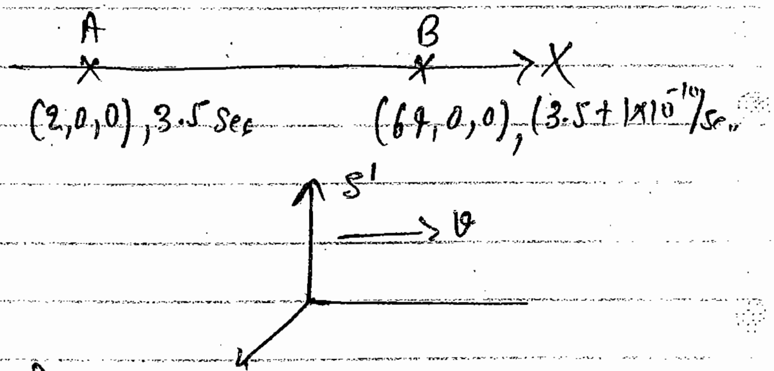
$$V_{\parallel} = V\cos 60° = \frac{\sqrt{3}}{4}c$$

$$AB = \frac{a\sqrt{13}}{4}$$

So Perimeter $= AB + BC + CA$

$$= \frac{a\sqrt{13}}{4} + \frac{a\sqrt{13}}{4} + \frac{a}{2} = \frac{2a\sqrt{13}}{4} + \frac{a}{2}$$

Q. 16



$$\Delta t' = \frac{\Delta t - \Delta x \cdot V/c^2}{\sqrt{1 - V^2/c^2}}$$

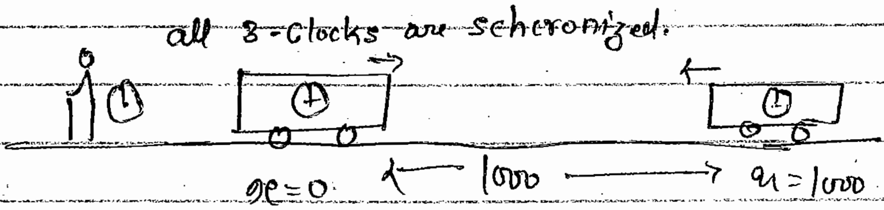
$$0 = \frac{\left(1 \times 10^{-10} - \frac{62 \times V}{c^2}\right)}{\sqrt{1 - V^2/c^2}}$$

$$\Rightarrow \frac{9 \times 10^{16} \times 10^{-10}}{62} = V$$

$$\Rightarrow V = \frac{9 \times 10^{6}}{62}$$

$$\Rightarrow \boxed{V = 1.5 \times 10^{5} \text{ m/se (A to B)}}$$

Q. 17



$$t' = \frac{t - xV/c^2}{\sqrt{1 - V^2/c^2}}$$

$$t_1' = \frac{0 - 0 \times V/c^2}{\sqrt{1 - V^2/c^2}} = 0$$

$$t_2' = \frac{0 - 1000 \times \frac{\left(-\frac{c}{2}\right)}{c^2}}{\sqrt{1-\frac{1}{4}}}$$

$$t_2' = \frac{500/c}{\sqrt{3}/2} = \frac{1000}{\sqrt{3}\,c}$$

Time difference b/w two clocks is $t_2' - t_1'$

$S' \longrightarrow S$ $\quad t = \dfrac{t' + x'v/c^2}{\sqrt{1 - v^2/c^2}}$ Observer is in S' (first measurement in S')

$S \longrightarrow S'$ $\quad t' = \dfrac{t - xv/c^2}{\sqrt{1 - v^2/c^2}}$ Observer is in S first measured in S

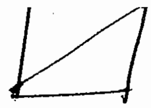
* Relativity of Geometrical Structure :-

(i) Length Contraction $\Rightarrow L = L_0\sqrt{1 - v_{\parallel}^2/c^2}$ | Result of measurement

(ii) Area Contraction :-

$$A = A_0\sqrt{1 - v_{\parallel}^2/c^2}$$

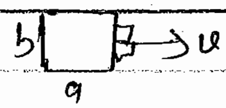
$v_{\parallel}$: Velocity $\parallel$el to surface.



some formula used.

(iii) Volume Contraction :-

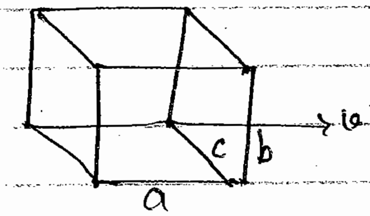
$$\boxed{V = V_0\sqrt{1 - v^2/c^2}}$$

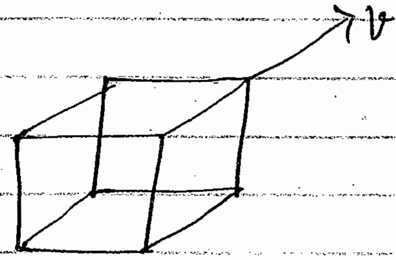


$A_0 = ab$

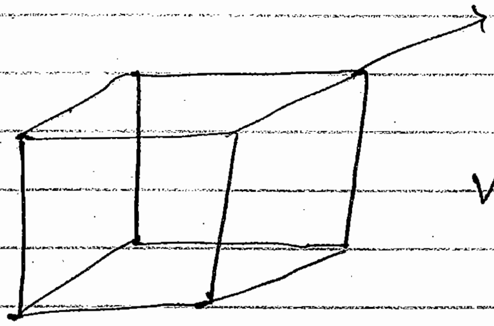
$A = (a\sqrt{1-v^2/c^2})\,b$

$A = A_0\sqrt{1 - v^2/c^2}$





Q.5



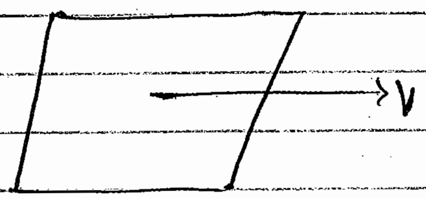
$V = V_0\sqrt{1 - v^2/c^2}$

$= a^3\sqrt{1 - \frac{16}{25}}$

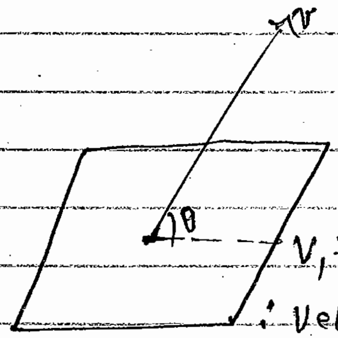
$$\boxed{V = \frac{3a^3}{5}}$$

$V = A \cdot h$

→



$V_{\parallel} = v$



$V_{\parallel} = V_{\parallel}\cos\theta$

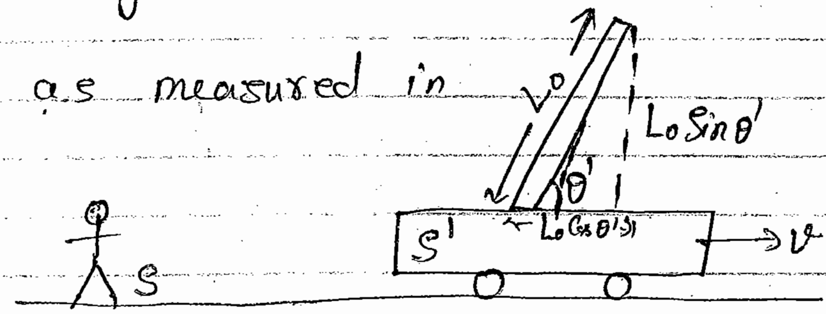
: velocity ∥ to surface.

⇒ But in case of volume velocity is directly put whatever direcⁿ it will move.

* Change of Geometrical Angle of an object:-

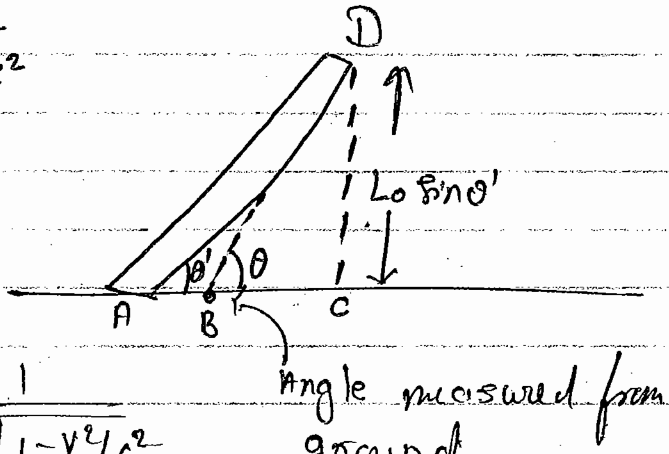
Let θ' = angle as measured in s'

θ = Angle as measured in s



$$\tan\theta = \frac{\cancel{L_0}\sin\theta'}{\cancel{L_0}\cos\theta'\sqrt{1-v^2/c^2}}$$

$$\boxed{\tan\theta = \frac{\tan\theta'}{\sqrt{1-v^2/c^2}}}$$



$\theta > \theta'$ $\quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

as $\gamma < 1$

- In relativity their is no concept of rigidity.

Here θ is the angle has been measured with direction of velocity.

$$\boxed{\theta = \tan^{-1}\gamma\tan\theta'}$$

Gate - 2011

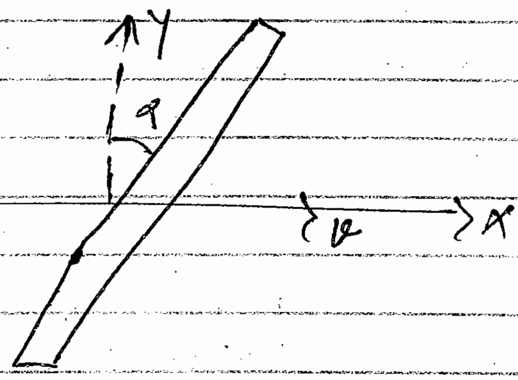
Q. A rod is moving along x-direction with speed v in the rest frame of the rod angle measured with y-axis is α. what is angle measured with y-axis in ground frame.

(a) $\tan^{-1}\gamma \tan\alpha$ ✓(b) $\tan^{-1}\dfrac{\tan\alpha}{\gamma}$

(c) $\tan^{-1}\gamma \cot\alpha$ (d) $\tan^{-1}\dfrac{1}{\gamma}\cot\alpha$.

<u>Soln</u>

Let β be the angle with y axis as seen from ground.



$$\theta' = 90 - \alpha$$

$$\theta = 90 - \beta$$

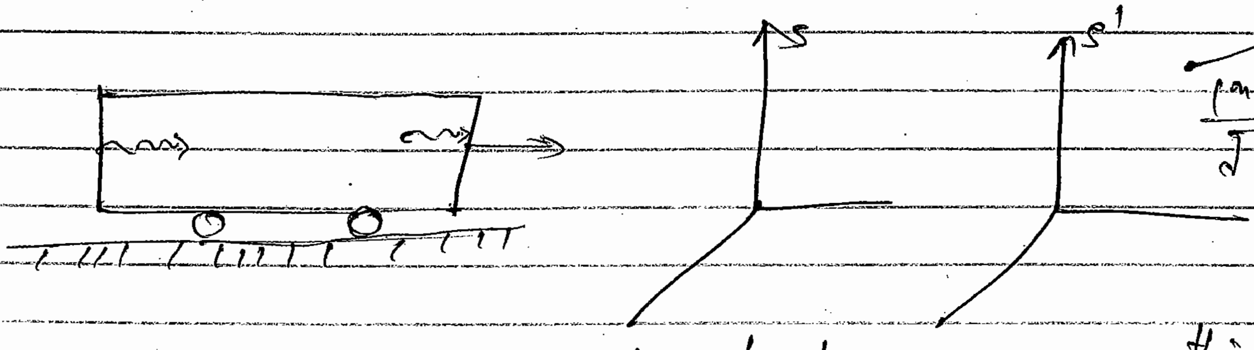
$$\therefore \boxed{\tan\theta = \gamma \tan\theta'}$$

So $\cot\beta = \gamma \cot\alpha$

$$\tan\beta = \frac{1}{\gamma}\tan\alpha$$

$$\boxed{\beta = \tan^{-1}\frac{1}{\gamma}\tan\alpha}$$

Note:



$$\frac{(ct + xv/c^2)}{\sqrt{1 - v^2/c^2}}$$

this is the co-ordinate of object not fram.

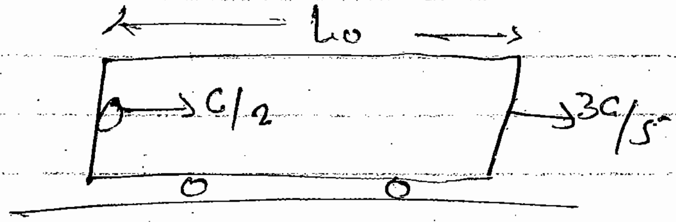
So we can not find length travelled by ou.

Q. {Related to previous Question.} Stone is ~~throwng~~ thrown from rear end of car to front end of car. find the distance travelled by stone.

Sol^n

Distance travelled by stone

$= \Delta x$



or = Velocity of stone w.r.to ground × time measured in ground.

Relativistic Dynamics

* Newtons Second Law :-

$$F = \frac{dp}{dt}$$ applicable when m is constant

$$= \frac{d}{dt}(mv)$$

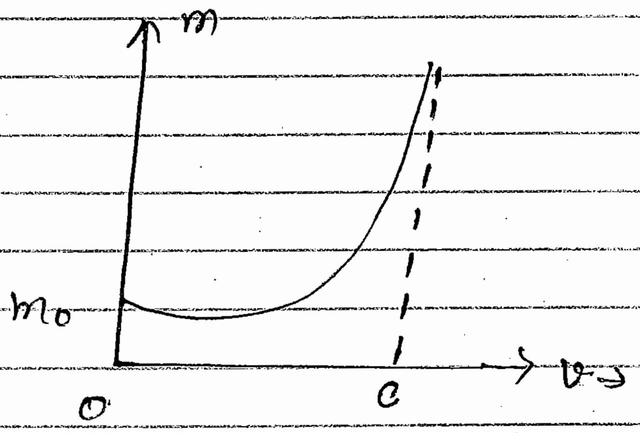
$\times$ $f = ma$ Not applicable when mass is variable.

* Variation of mass with velocity :-

Rest mass.

$$\boxed{m = \frac{m_0}{\sqrt{1 - v^2/c^2}}} , \; m_0 \neq 0$$

Dynamic mass.



$$m\sqrt{1 - v^2/c^2} = m_0$$

If dynamic mass is not zero but rest mass is 0 then

$$m\sqrt{1 - v^2/c^2} = 0$$

$$1 - v^2/c^2 = 0$$

$$\boxed{v = c}$$

* If rest mass of a particle is zero then

it must move with speed of light.
e.g. Newtrino.

$$\nu \longrightarrow v = c$$

$$m_0 = 0$$

* Momentum :-

$$p = mv$$

$$\boxed{p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}} \qquad m_0 \neq 0$$

* Kinetic Energy :-

K = Total Energy − Rest Energy

$$K = E - E_0$$

$$\boxed{K = mc^2 - m_0c^2}$$

In limit '$v \ll c$' then — $K = \frac{1}{2} m_0 v^2$

$$K = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0c^2$$

$$\boxed{E = mc^2} \longrightarrow \text{Total Energy (K.E. + R.E.)} = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}$$

* Kinetic Energy and Momentum Relation :-

$$pc = \sqrt{K(K + 2m_0c^2)}$$

* Total Energy and Momentum Relation :-

$$\boxed{E = \sqrt{p^2c^2 + m_0^2c^4}}$$

* for $m_0 = 0$

$$\boxed{p = \frac{E}{c}}$$

* for Non - Relativistic Case :-

$$\boxed{E = K = \frac{1}{2} m_0 v^2}$$

or

$$\boxed{E = K = \frac{p^2}{2m}}$$

Q. Kinetic Energy of a particle is $2m_0c^2$ what is it's momentum.

Sol^n

$$p = \sqrt{K(K + 2m_0c^2)}$$

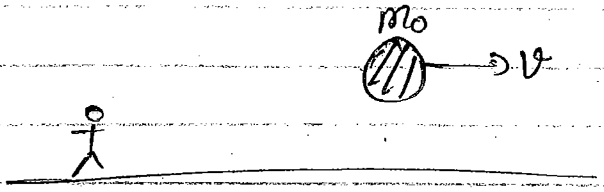
$$= \sqrt{2m_0c^2(2m_0c^2 + 2m_0c^2)}$$

$$= \sqrt{8m_0^2c^4}$$

$$\boxed{p = 2\sqrt{2}\, m_0 c}$$ Ans

* Important Note :-

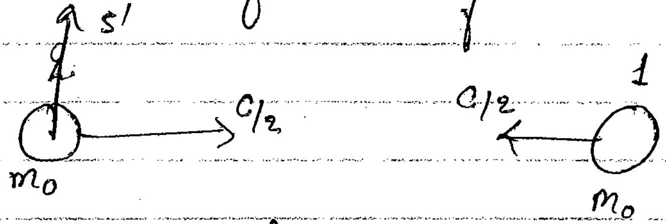
(1) $$m = \frac{m_0}{\sqrt{1 - V^2/c^2}}$$



V = Relative velocity in a frame in which mass is being measured.

Q. 2 e^{-}s of rest mass m_0 are moving towards each other with speed $c/2$ as seen from lab frame. What is dynamic mass of one e^{-} in the rest frame of another e^{-}.

Sol^n



Let us calculate mass of 1 in rest frame of 2 (w.r.to. 2.)

$$m_1 = \frac{m_0}{\sqrt{1 - \frac{V_{12}^2}{c^2}}}$$

$$V_{12} = V_{1}' = \frac{V_1 - V}{1 - V_1 V/c^2}$$

$$= \frac{-\frac{c}{2} - \frac{c}{2}}{1 + \frac{1}{4}} = \frac{-4c}{5}$$

So $$m_1 = \frac{m_0}{\sqrt{1 - \frac{16}{25}}} = \frac{5m_0}{3}$$

$$\boxed{m_1 = \frac{5m_0}{3}}$$

* Relativistic Collision / Breaking :-

* In all relativistic Case :-

(i) Momentum of the System is conserved.

$$\boxed{\vec{P_i} = \vec{P_f}}$$

Identification :- Words used
→ $v = ?c$
→ massless particle
→ rest mass then collision is relativistic.

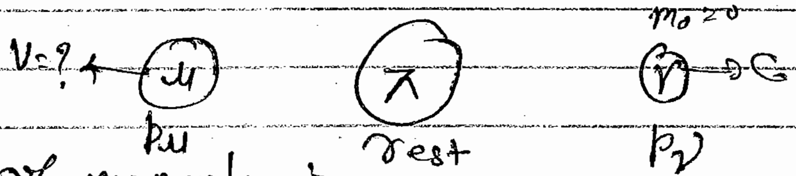
(ii) Total energy is conserved.

$$\boxed{E_{initial} = E_{final}}$$

Q.58 A pion of mass m_π (Rest mass) at rest decays into a muon of mass m_μ (rest mass) and a neutrino of zero mass. Speed of muon is ?

Sol^n

$$\pi \longrightarrow \mu + \nu$$



Conservation of momentum :-

$\because P_{initial} = 0$

So $p_\mu = p_\nu$ (opposite in direc^n.)

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

Let v = speed of muon.

$$\frac{m_\mu v}{\sqrt{1 - \frac{v^2}{c^2}}} = p_\nu$$

$$\frac{m_\mu v}{\sqrt{1 - v^2/c^2}} = \frac{E_\nu}{c} \quad \left\{ \because \text{rest mass is zero} \right\}$$

So $$E_\nu = \frac{m_\mu v c}{\sqrt{1 - v^2/c^2}}$$

Apply Energy Conservation:-

$$E_{initial} = E_{final}$$

Rest Energy $\longrightarrow$ $$m_\pi c^2 = E_\mu + E_\nu$$

$$m_\pi c^2 = \frac{m_\mu c^2}{\sqrt{1 - v^2/c^2}} + \frac{m_\mu v c}{\sqrt{1 - v^2/c^2}}$$

$$m_\pi \cancel{c^2} = \frac{m_\mu \cancel{c^2}}{\sqrt{1 - v^2/c^2}} \left[1 + \frac{v}{c} \right]$$

$$\frac{m_\pi}{m_\mu} = \frac{(1 + v/c)}{\sqrt{1 - \frac{v}{c}}\sqrt{1 + \frac{v}{c}}}$$

$$\frac{m_\pi}{m_\mu} = \sqrt{\frac{1 + v/c}{1 - v/c}}$$

$$\frac{m_\pi^2}{m_\mu^2} = \frac{1 + v/c}{1 - v/c}$$

Componendo-dividendo

$$\frac{m_\pi^2 - m_\mu^2}{m_\mu^2}$$

AE T 2010-11

Dated :- 17/Sep/2014

(61)

Decay Process :-

$$\tau^- \longrightarrow \pi^- + \nu$$

rest

$$m_0 = M_\tau \qquad M_\pi \qquad 0$$

$E_\pi = ?$

$\leftarrow \pi$ p_π $\qquad$ τ^- Rest $\qquad$ $\nu \longrightarrow c$ p_ν

$$p_\pi = p_\nu \quad \text{——— (1)}$$

Let E_π is energy of pion.
from conservation of Energy -

$$E_\tau = \underline{E_\pi} + \underline{E_\nu}$$

$$M_\tau c^2 = E_\pi + p_\nu c$$

$$\Rightarrow M_\tau c^2 = E_\pi + \sqrt{E_\pi^2 - m_\pi^2 c^4}$$

$$\Rightarrow (M_\tau c^2 - E_\pi)^2 = E_\pi^2 - m_\pi^2 c^4$$

$$\Rightarrow M_\tau^2 c^4 + \cancel{E_\pi^2} - 2M_\tau c^2 E_\pi = \cancel{E_\pi^2} - m_\pi^2 c^4$$

$$\Rightarrow c^{\cancel{4}}(M_\tau^2 + M_\pi^2) = 2M_\tau \cancel{c^2} E_\pi$$

$$\Rightarrow \boxed{E_\pi = \frac{(M_\tau^2 + M_\pi^2)c^2}{2M_\tau}}$$

Ans

$$E_\nu = p_\nu c$$

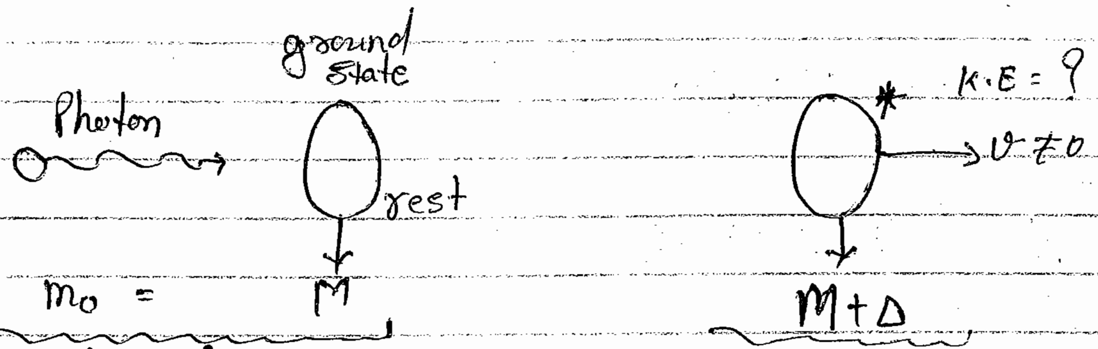
$$E_\pi = \sqrt{p_\pi^2 c^2 + M_\pi^2 c^4}$$

$$E_\pi = \sqrt{p_\nu^2 c^2 + M_\pi^2 c^4}$$

$$\sqrt{\frac{E_\pi^2 - m_\pi^2 c^4}{c^2}} = p_\nu$$

Total Energy $\rightarrow E = K + m_0c^2$

Q.60



initial
Before absorption

After Absorption
final.

Conservation of momentum :-

$$p_{initial} = p_{final}$$

$$\Rightarrow p_{photon} + p_{atom} = p^*_{atom}$$

$$\Rightarrow \frac{E_p}{c} + 0 = p^*_{atom}$$

$$\frac{E_p}{c} = p_a \quad \text{—— (1)}$$

Conservation of energy :-

$$E_{photon} + E_{atom} = E^*_{atom}$$

$$E_p + Mc^2 = E^*_{atom} \quad \text{—— (2)}$$

$$\Rightarrow p_a c + Mc^2 = E^*_{atom} = K + m_0c^2$$

$$\Rightarrow p_a c + Mc^2 = K + (M+\Delta)c^2$$

$$\therefore \; p_c = \sqrt{K(K + 2m_0c^2)}$$

So ~~$p_a c$~~

$$\sqrt{K(K+2m_0c^2)} + Mc^2 = K + (M+\Delta)c^2$$

$$\Rightarrow \sqrt{K[K+2(M+\Delta)c^2]} = K + \cancel{Mc^2} + \Delta c^2 - \cancel{Mc^2}$$

$$\Rightarrow \sqrt{K[K+2(M+\Delta)c^2} = K+\Delta c^2$$

Squiring -

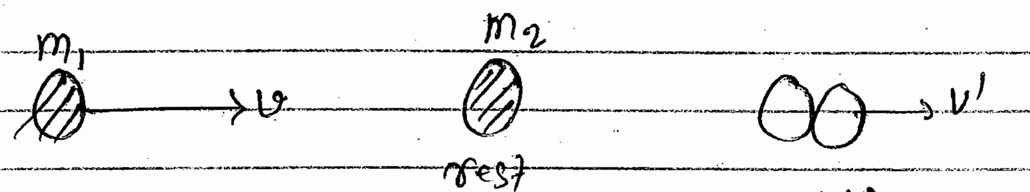
$$\cancel{K^2} + 2K(M+\Delta)c^2 = \cancel{K^2} + \Delta^2c^4 + 2K\Delta c^2$$

$$2Mc^2K + \cancel{2\Delta c^2 K} = \Delta^2c^4 + \cancel{2K\Delta c^2}$$

$$\boxed{K = \frac{\Delta^2c^2}{2M}}$$

Q. A particle of rest mass m_1 moving with speed v collides with another particle of mass m_2 which is initially at rest and sticks to it. What is final velocity of the new system and what is new rest mass?

Sol^n



$m_0 = M \geq m_1 + m_2$

Let v' = final speed of combined masses

M - final rest mass of " "

$\Rightarrow P_{initial} = P_{final}$

$$\frac{m_1 v}{\sqrt{1-v^2/c^2}} + 0 = \frac{M v'}{\sqrt{1-v'^2/c^2}} \quad \text{--- (1)}$$

$$\frac{m_1 v}{\sqrt{1-v^2/c^2}} = \frac{M v'}{\sqrt{1-v'^2/c^2}} \quad \text{(1)}$$

Conservation of energy -

$$E_{initial} = E_{final}$$

$$\frac{m_1 c^2}{\sqrt{1-v^2/c^2}} + m_2 c^2 = \frac{M c^2}{\sqrt{1-v'^2/c^2}} \quad \text{(2)}$$

Divide (I)/(II) -

$$\left[\frac{\frac{m_1 c^2}{\sqrt{1-v^2/c^2}} + m_2 c^2}{\frac{m_1 v}{\sqrt{1-v^2/c^2}}}\right]^{-1} = \frac{v'}{c^2}$$

$$\frac{v'}{c^2} = \left[\frac{c^2}{v} + \frac{m_2 c^2}{m_1 v} \times \sqrt{1-v^2/c^2}\right]^{-1}$$

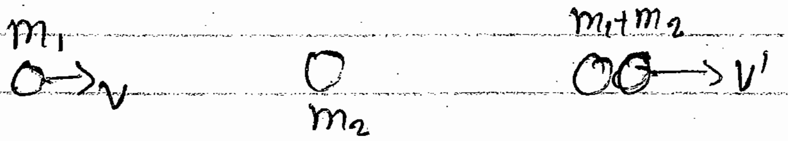
$$\frac{v'}{\cancel{c^2}} = \frac{1}{\frac{\cancel{c^2}}{v}\left[1 + m_2/m_1 \times \sqrt{1-v^2/c^2}\right]}$$

$$\boxed{v' = \frac{v}{1 + \frac{m_2}{m_1}\sqrt{1-v^2/c^2}}}$$

For non Relativistic :- $v << c \Rightarrow \sqrt{1-v^2/c^2} \approx 1$

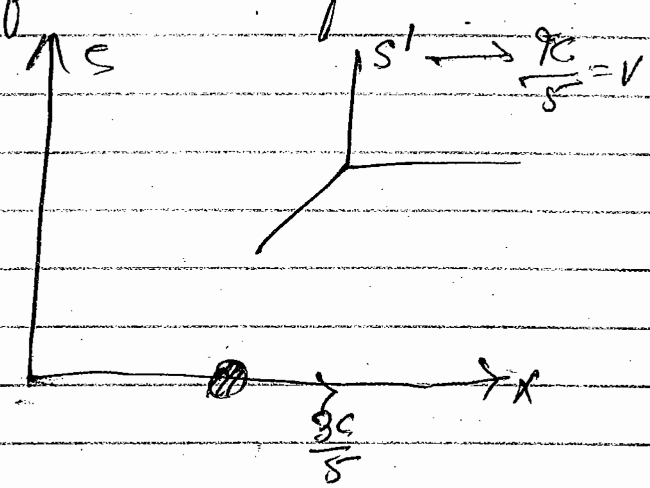
$$\boxed{v' = \frac{m_1 v}{m_1 + m_2}}$$

* Non Relativistic Case :-



$$\boxed{m_1 v = (m_1+m_2) v'}$$

Q. An electron of rest mass m_0 is moving in x-direc^n with velocity $\frac{3c}{5}$ in S frame. What is the energy of $\bar{e}^5$ in S' frame.



$$E = mc^2$$

$$= \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}$$

Energy in S' frame.

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v_x'^2}{c^2}}}$$

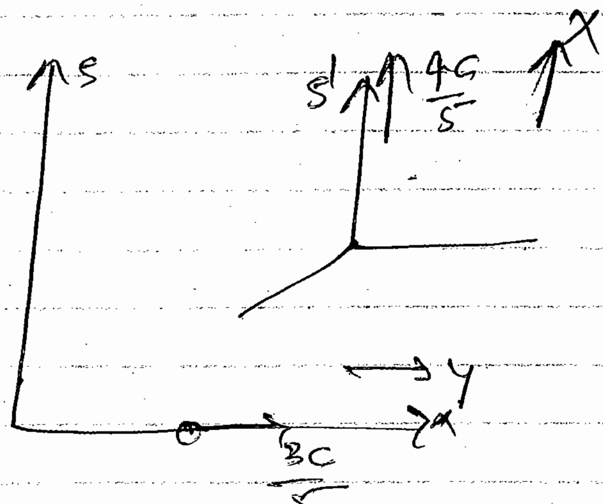
$$v_x' = \frac{v_x - V}{1 - \frac{v_x V}{c^2}}$$

$$= \frac{\frac{4c}{5} - \frac{3c}{5}}{1 - \frac{\frac{3c \cdot 4c}{5 \cdot 5}}{c^2}}$$

$$= \frac{5}{13} c$$

Q. In previous question what is energy of e^- in S' frame where S' frame is moving in y direc^n

$\underline{Sol^n}$ ∵ Direc^n of movement of S' frame is considered as x-direc^n. So particle is moving in y direc^n.



$V_y = \frac{3c}{5}$

$V_x = 0$

$$E' = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$V_x' = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{V_x - V}{1 - \frac{V_x V}{c^2}}$$

$$= \frac{0 - \frac{4c}{5}}{1-0} = -\frac{4c}{5}$$

$$\boxed{V_x' = -\frac{4c}{5}}$$

$$V_y' = \frac{V_y\sqrt{1-v^2/c^2}}{1 - \frac{V_x \cdot V}{c^2}} = \frac{3c}{5} \times \frac{3}{5} = \frac{9c}{25}$$

$$V' = \sqrt{V_x'^2 + V_y'^2} = \sqrt{\frac{16}{25}c^2 + \frac{81}{625}c^2}$$

$$\boxed{V' = \sqrt{\frac{481}{625}c^2}}$$

Then $E' = \frac{m_0 c^2}{\sqrt{1 - \frac{481}{625}}}$

$$\boxed{E' = \frac{25}{12} m_0 c^2}$$

Q. A photon of wavelength λ strikes an electron of rest mass m_0 which is initially at rest. If the photon is scattered by e^- in opposite direcn. what is new wavelength of photon and what is K.E. of e^-.

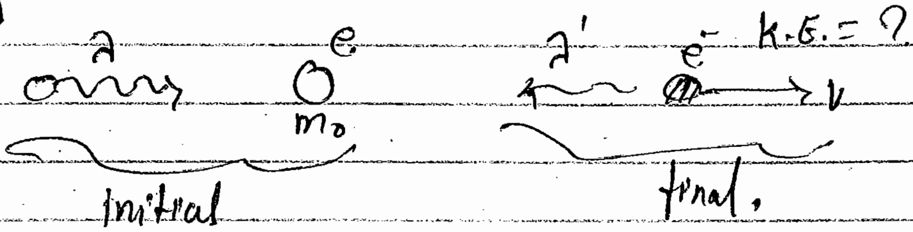
Sol^n

$$\lambda' = \lambda + \frac{h}{m_0 c}(1 - \cos\theta)$$

$$\lambda' = \lambda + \frac{2h}{m_0 c}$$

$$\boxed{\lambda' - \lambda = \frac{2h}{m_0 c}}$$

K.E. of electron :-



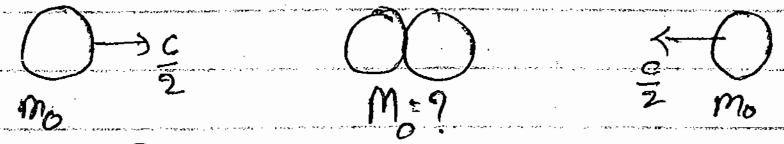
$$T.E_{initial} = T.E._{final}$$

$$\frac{hc}{\lambda} + \cancel{m_0 c^2} = \frac{hc}{\lambda'} + k + \cancel{m_0 c^2}$$

$$\boxed{k = hc\left[\frac{1}{\lambda} - \frac{1}{\lambda'}\right]}$$

Q. Two particles of equal rest mass m_0 are moving in opposite direc^n with speed $c/2$. After collision they strick each other what is new rest mass?

Sol^n



(a) $\frac{4m_0}{\sqrt{3}}$ (b) $\frac{2m_0}{\sqrt{3}}$ (c) $2m_0$ (d) $\frac{5m_0}{\sqrt{3}}$

$$P_{initial} = P_{final}$$

$$\cancel{\frac{m_0 \cdot \frac{c}{2}}{\sqrt{1-\frac{1}{4}}}} - \cancel{\frac{m_0 \cdot c/2}{\sqrt{1-\frac{1}{4}}}} = P_{final}$$

$$\boxed{P_{final} = 0}$$

So final mass is at rest.

$$T.E_i = T.E_f$$

$$\frac{m_0}{\sqrt{1-\frac{1}{4}}}c^2 + \frac{m_0 c^2}{\sqrt{1-\frac{1}{4}}} = M_0 c^2$$

$$\frac{4m_0 \cancel{c^2}}{\sqrt{3}} = M_0 \cancel{c^2}$$

$$\boxed{M_0 = \frac{4}{\sqrt{3}} m_0}$$

$$\boxed{M_0 > 2m_0}$$

Q. A particle of rest mass m_0 ~~is~~ has momentum $2m_0c$ what is total energy of the particle and what is its speed.

Sol

$$E = \sqrt{p^2c^2 + m_0^2c^4}$$

$$= \sqrt{5 m_0^2 c^4}$$

$$\boxed{E = \sqrt{5}\, m_0 c^2}$$

$\because$ $p = mv$

and $E = mc^2$

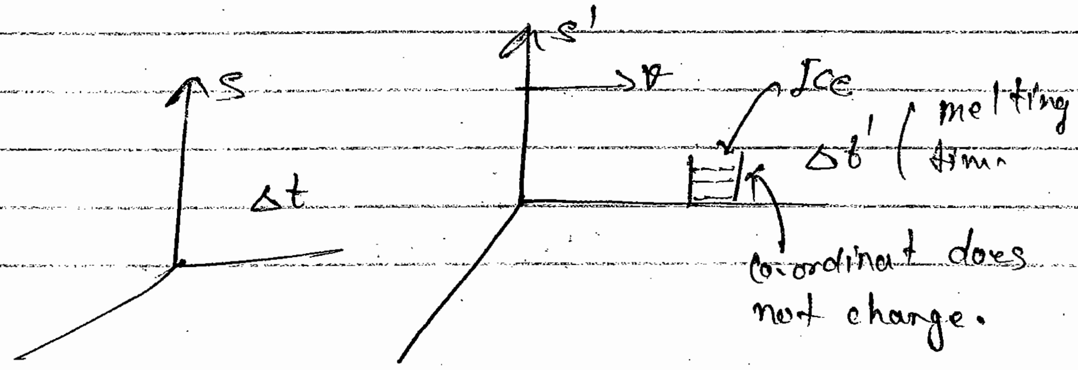
So $\frac{p}{E} = \frac{v}{c^2}$

So $\boxed{v = \frac{p}{E} c^2}$

$$\boxed{v = \frac{2}{\sqrt{5}} c}$$

* Time Dilation :-

All process (may be Physical, Chemical and biological) slow down while in motion. Due to which completion of process takes more time.



$\Delta x' = 0$

but $\Delta x \neq 0$

so $\Delta t = \dfrac{\Delta t' + V\Delta x'/c^2}{\sqrt{1 - V^2/c^2}}$

$$\boxed{\Delta t = \frac{\Delta t'}{\sqrt{1 - V^2/c^2}}} \quad \{\because \Delta x' = 0$$

Relation b/w time duration.

$\Delta t'$ = time measured in the frame in which sample is at rest (proper time) {must $\Delta x' = 0$}

Δt = is time measure in the frame in which it is moving.

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - V^2/c^2}}$$

Proper life time of particle

Life time measured in Lab frame. $\boxed{\Delta t > \Delta t'}$

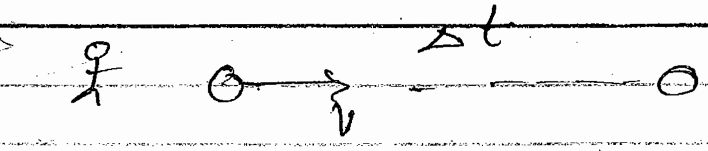
Q. A particle of rest mass m_0 has proper time 2×10^{-6} sec {2 μs.} if total energy of the particle is $2m_0c^2$ what distance it will travell during its lifetime,

Sol^n

$\Delta t' = 2 \times 10^{-6}$ sec

m_0

$E = 2m_0c^2$



Distance = velocity × time (Δt)

$$= v \cdot \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}$$

$$E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}$$

$$2\cancel{m_0 c^2} = \frac{\cancel{m_0 c^2}}{\sqrt{1 - v^2/c^2}}$$

$$1 - \frac{v^2}{c^2} = \frac{1}{4}$$

$$v = \frac{\sqrt{3}}{2} c$$

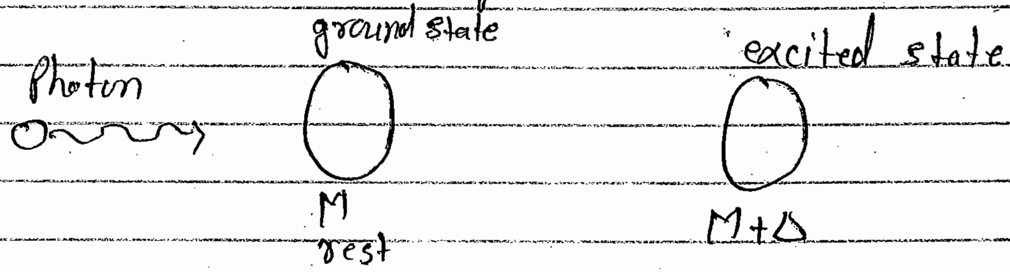
So Distance $= \frac{\sqrt{3}}{2} c \cdot \frac{\Delta t'}{\sqrt{1 - \frac{3}{4}}}$

$$= \frac{\sqrt{3}}{\cancel{2}} \, 3 \times 10^8 \times 2 \times 10^{-6} \cancel{\times 2}$$

$$= 6\sqrt{3} \times 10^2 \text{ meter.}$$

2011/12 V.E.T

Q. What is frequency of photon?



Sol^n Applying conservation of momentum -

let energy of photon is E_p. $\left\{ \text{momentum} = \frac{E_p}{c} \right\}$

$$p_{initial} = p_{final}$$

$$\frac{E_p}{c} + 0 = p_{final} = p_{atom}$$

$$E_p = c\, p_{atom} = c\, p_a \quad (1)$$

← momentum of atom in excited state

Applying conservation of Energy -

$$T.E_i = TE_f.$$

$$E_p + Mc^2 = \sqrt{P_a^2 c^2 + (M+\Delta)^2 c^4}$$

$$= \sqrt{\left(\frac{E_p}{\cancel{c}}\right)^2 \cancel{c^2} + (M+\Delta)^2 c^4}$$

$$(E_p + Mc^2) = \sqrt{E_p^2 + (M+\Delta)^2 c^4}$$

Squiring on both side -

$$(E_p + Mc^2)^2 = E_p^2 + (M+\Delta)^2 c^4$$

$$\cancel{E_p^2} + \cancel{M^2 c^4} + 2E_p M c^2 = \cancel{E_p^2} + \cancel{M^2 c^4} + \Delta^2 c^4 + 2M\Delta c^4$$

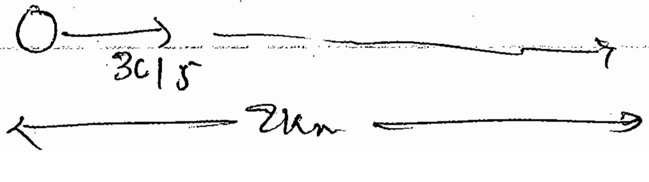
$$E_p = \frac{c^4(\Delta^2 + 2M\Delta)}{2Mc^2}$$

$$E_p = \frac{(\Delta^2 + 2M\Delta)c^2}{2M}$$

$$\frac{E_p}{h} =$$

Q. A particle (unstable) moving with speed $\frac{3c}{5}$ can travell the maximum distance of 2km. as measure in lab from what is proper life-time of the particle.

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - V^2/c^2}}$$



Distance = velocity × time

$$2 \times 10^3 = \frac{3}{5}c \times \Delta t$$

$$2 \times 10^3 = \frac{3}{5}c \times \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{3}{5}c \times \frac{\Delta t'}{\frac{4}{5}}$$

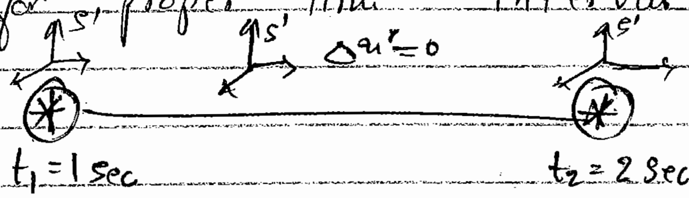
$$\text{so } \Delta t' = \frac{8 \times 10^3}{3c} = \frac{8 \times 10^3}{3 \times 3 \times 10^8}$$

$$= \frac{8}{9} \times 10^{-5}\ \text{sec}$$

$$\boxed{\Delta t' = 8.8\ \mu\text{sec}}$$ Ans

2. The two events take place in lab frame at points $(2,0,0)$, $(22,0,0)$ at times $t_1 = 1$ sec and $t_2 = 2$ sec respectively. Is the time interval measured for these two events is proper time interval. What is velocity of a frame moving along the line joining the two events in which time interval is proper.

Sol^n No, it is not a proper time interval becoz for proper time interval $\Delta x = 0$



$$\Delta x' = \frac{\Delta x - V\Delta t}{\sqrt{1 - V^2/c^2}}$$

$$0 = \frac{20 - V.1}{\sqrt{1 - V^2/c^2}}$$

$$\boxed{V = 20 \text{ m/sec}}$$

Proper time interval $= \Delta t'$

$$\Delta t' = \frac{\Delta t - \Delta x . V/c^2}{\sqrt{1 - V^2/c^2}}$$

$$= \frac{1 - 20.20/c^2}{\sqrt{1 - 400/c^2}}$$

$$\Delta t' = \frac{1 - 400/c^2}{\sqrt{1 - \frac{400}{c^2}}} = \sqrt{1 - 400/c^2}$$

$$\boxed{\Delta t' = \sqrt{1 - 400/c^2}}$$

From time dilation formula:-

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - V^2/c^2}}$$

$$1 = \frac{\Delta t'}{\sqrt{1 - \frac{400}{c^2}}}$$

$$\boxed{\Delta t' = \sqrt{1 - \frac{400}{c^2}}}$$

By L. T. :-

$$\Delta x' = (\Delta x - v\Delta t)\gamma \quad \text{---} (1)$$

$$\Delta t' = \left(\Delta t - \frac{\Delta x . v}{c^2}\right)\gamma \quad \text{---} (2)$$

$$\Delta y' = \Delta y \quad \text{---} (3)$$

$$\Delta z' = \Delta z \quad \text{---} (4)$$

$(1)^2 + (3)^2 + (4)^2 - c(2)^2$:-

$$\Delta x'^2 + \Delta y'^2 + \Delta z'^2 - c^2\Delta t'^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2$$

for movement along x-direc^n :-

$$\boxed{\Delta x'^2 - c^2\Delta t'^2 = \Delta x^2 - c^2\Delta t^2}$$

By putting the value of Δx and Δt we get proper time interval.

* Lorentz Invariant quantities :-

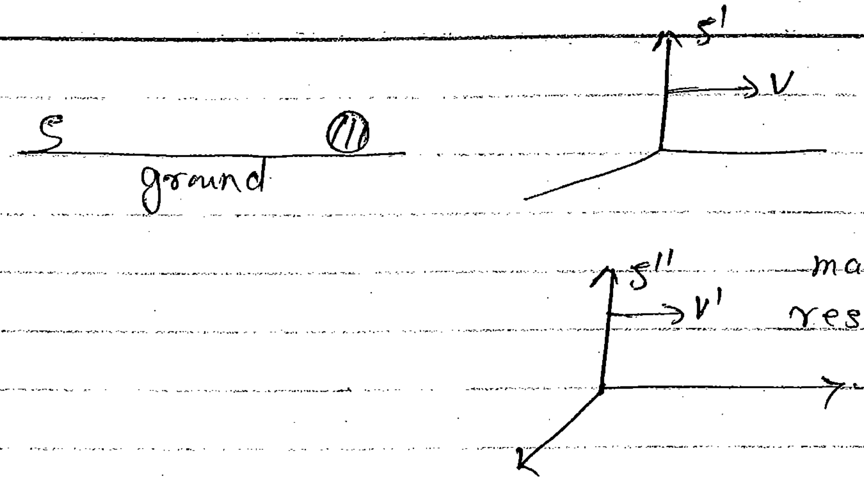
What it means is that the value of quantity will be same in S and S' frame!

(i) $\underbrace{\Delta r'^2 - c^2\Delta t'^2}$ $= \Delta r^2 - c^2\Delta t^2$

$$dS^2 = \Delta r^2 - c^2\Delta t^2 \Longrightarrow$$ Distance in space-time (Distance in Minkowski space)

Distance in space time remains invariant under Lorentz transformation

(ii) Rest mass, proper time, proper length.



mass = m

rest mass $= m\sqrt{1-v^2/c^2}$

mass = m'

rest mass $= m'\sqrt{1-v'^2/c^2}$

(iii) Charge, Maxwell Equations are invariant.
$\vec{E}\cdot\vec{B}$, $E^2 - c^2B^2$

Charge density $= \dfrac{\text{Charge}}{\text{Volume}}$

$$\rho_0 = \frac{Q_0}{V_0}$$

$$\rho = \frac{Q_0}{V} = \frac{Q_0}{V_0\sqrt{1-v^2/c^2}}$$

$$\boxed{\rho = \frac{\rho_0}{\sqrt{1-v^2/c^2}}}$$

Charge density is not ~~in~~ Lorentz invariant.

* $E^2 - c^2p^2$ is also Lorentz invariant:-

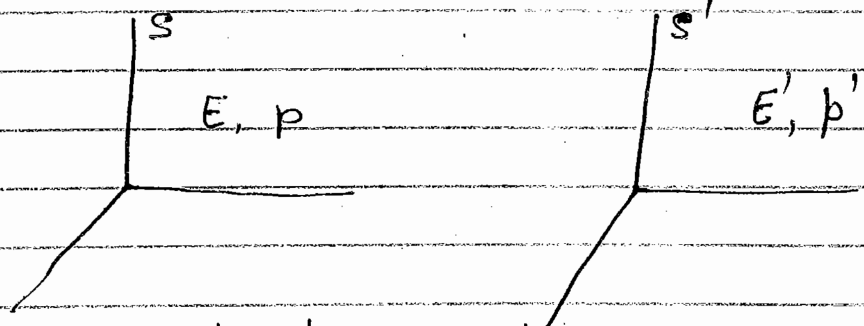
$$E = \sqrt{p^2c^2 + m_0^2c^4}$$

$$E^2 - p^2c^2 = \underbrace{m_0^2c^4}_{\text{Lorentz Invariant}}$$

So it also.

Taken by :- J. D. Jackson {Electrodynamics}.

* Energy Momentum Transformation Under Lorentz Transformation :-



Let E and p is the energy and momentum in S
and E' and p' is the energy and momentum in S'.

$$E = mc^2$$

$$= \frac{m_0 c^2}{\sqrt{1 - V^2/c^2}} \times \frac{dt}{dt} \qquad \left\{ dt = \frac{dt'}{\sqrt{1-V^2/c^2}} \right\}$$

$$E = \frac{m_0 c^2}{dt'} dt \qquad \left\{ dt = \frac{dt'}{m_0} \frac{E}{c^2} \right.$$

Proper time.

Invariant under Lorentz transformation.

$$E \propto dt$$

So energy is transforms as same way time transforms.
E will transform in same way as 'dt' does.

Momentum :-

$$p_x = m v_x$$

$$= \frac{m_0}{\sqrt{1 - \frac{V^2}{c^2}}} \times \frac{dx}{dt}$$

$$p = \frac{m_0}{dt'} dx \qquad dx = \frac{dt'}{m_0} p$$

Invariant under lorentz transformation

So we can say p will transforme in same way as 'dx' does.

* Energy Transformation Relation :-

$$\because \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - v^2/c^2}}$$

$t \to k\frac{E'}{c^2}$, $x \to k\,p$. when $K = \frac{dt'}{m_0}$ ← Invariant.

So

$$\frac{E'}{c^2} = \frac{\frac{E}{c^2} - \frac{v p_x}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\boxed{E' = \frac{E - p_x V}{\sqrt{1 - \frac{v^2}{c^2}}}} \leftarrow \text{JEST}$$

* Momentum Transformation Relation :-

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

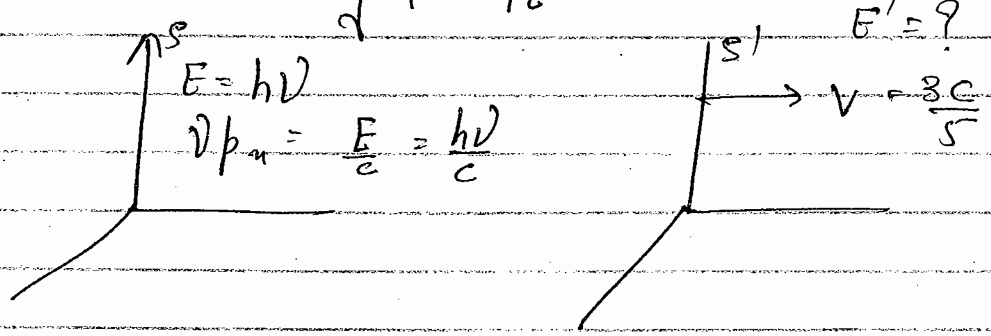
$$\boxed{p_x' = \frac{p_x - V.\frac{E}{c^2}}{\sqrt{1 - v^2/c^2}}}$$

$$\left(E' = \frac{m_0 c^2}{\sqrt{1 - v_x^2/c^2}} \right)$$ ← Not applicaple in case of photon becoz it rest mass is zero, and v is sam in all fram.

Prob: A photon of frequency ν is moving along x-direcn in S frame. what is energy of photon in S' frame which is also moving in x-direcn with speed $\frac{3c}{5}$.

Soln

$$E' = \frac{E - p_x v}{\sqrt{1 - v^2/c^2}}$$

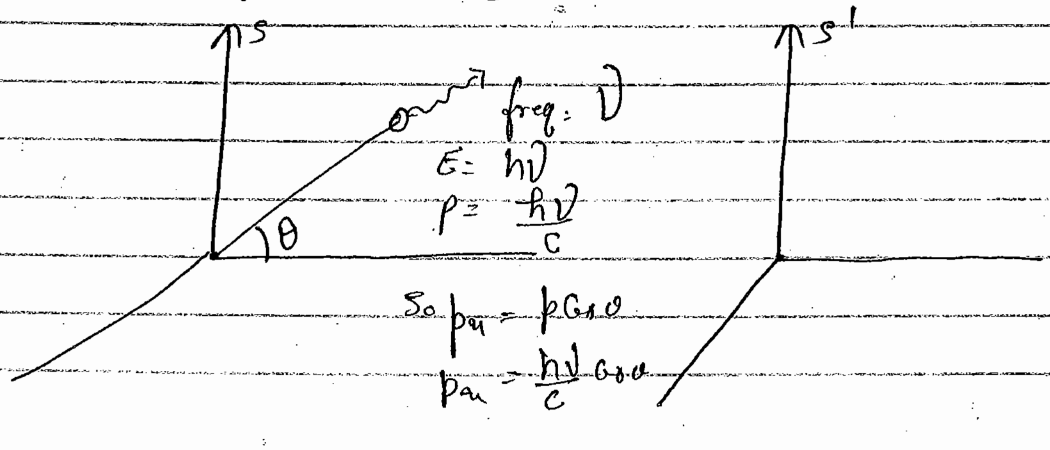


So
$$E' = \frac{h\nu - \frac{h\nu}{c}\cdot\frac{3c}{5}}{4/5} = \frac{h\nu - \frac{3h\nu}{5}}{4/5}$$

$$\boxed{E' = \frac{h\nu}{2}}$$

Q. In Lab frame a photon is moving at angle θ in x-direcn if its freq. in lab frame is ν. What is freq. of photon in S' frame which is moving with speed $\frac{4c}{5}$ in x-direcn.

Soln



So $p_x = p\cos\theta$

$p_x = \frac{h\nu}{c}\cos\theta$

Last two question is like a Doppler's Effect.

$$E' = \frac{E - p_x v}{\sqrt{1 - v^2/c^2}} = \frac{h\nu - \frac{h\nu}{c}\cos\theta \frac{4c}{5}}{3/5}$$

$$E' = \frac{h\left(1 - \frac{4}{5}\cos\theta\right)\nu}{3/5} = \frac{h\nu(5 - 4\cos\theta)}{3}$$

$\therefore \quad \nu' = \frac{E'}{h}$

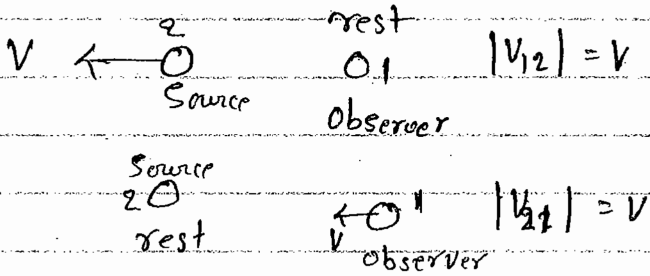
So $\boxed{\nu' = \frac{\nu(5 - 4\cos\theta)}{3}}$ Ans

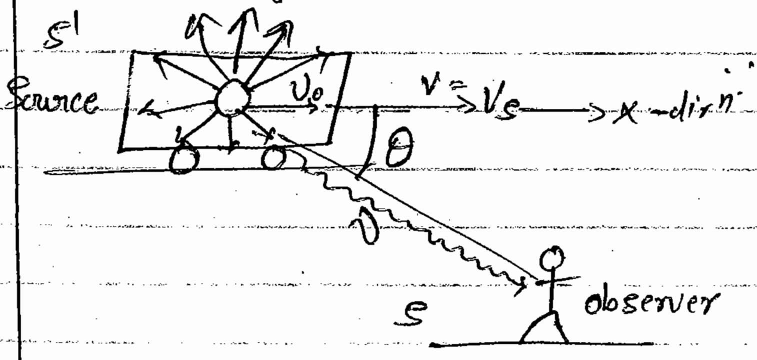
* Doppler Effect of Light:-

The phenomenon of change in frequency of light due to relative velocity or motion.

Let ν_0 is natural frequency of light emitted by ~~sound~~ source which is moving with speed V_s.

ν_0 is frequency measured in rest frame of source.





$|V_{12}| = |V_{21}|$

$\boxed{\nu_1 = \nu_2}$

Let ν = frequency in S frame (in observer frame)
θ = angle b/w dirn of V_s and line joining the source and observer.

In case of Doppler

$$E' = \frac{E - p_x \cdot V}{\sqrt{1 - V^2/c^2}}$$

$$h\nu_0 = \frac{h\nu - \frac{h\nu}{c}\cos\theta \cdot V_s}{\sqrt{1 - V_s^2/c^2}}$$

$$\nu_0 = \frac{\nu - \frac{\nu}{c}\cos\theta \cdot V_s}{\sqrt{1 - V_s^2/c^2}}$$

So $$\boxed{\nu = \frac{\nu_0 \sqrt{1 - V_s^2/c^2}}{1 - \frac{V_s}{c}\cos\theta}}$$

Special Case :-

When $\theta = 0$

$$\nu = \frac{\nu_0 \sqrt{1 - V_s^2/c^2}}{1 - V_s/c}$$

$$\boxed{\nu = \nu_0 \sqrt{\frac{1 + V_s/c}{1 - V_s/c}}} \qquad \nu > \nu_0$$

If distance b/w source and observer is decreasing then $\nu > \nu_0$ and vice-versa.

When $\theta = 180°$

$$\boxed{\nu = \nu_0 \sqrt{\frac{1 - V_s/c}{1 + V_s/c}}} \qquad \nu < \nu_0$$

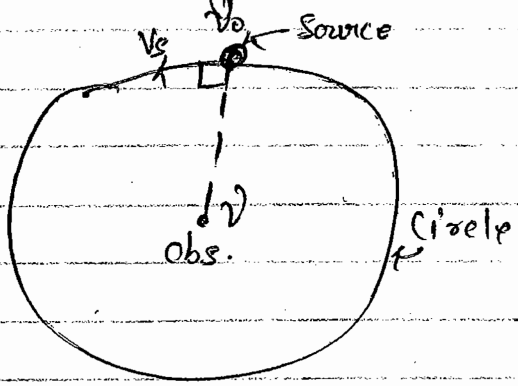
* If both Source and observer is moving:-

$$\boxed{\nu = \nu_0 \frac{\sqrt{1 - v^2_{relativistic}/c^2}}{1 - \frac{v_{relativistic}}{c} \cdot \cos\theta}}$$

$\theta = 90^\circ$

$$\boxed{\nu = \nu_0 \sqrt{1 - v_s^2/c^2}}$$ ← Transverse Doppler's Effect.

In case of sound no Transverse Doppler's Effect because at $\theta = 90^\circ$ $\nu = \nu_0$ in sound but in light $\nu \neq \nu_0$ at $\theta = 90^\circ$.



In case of sound

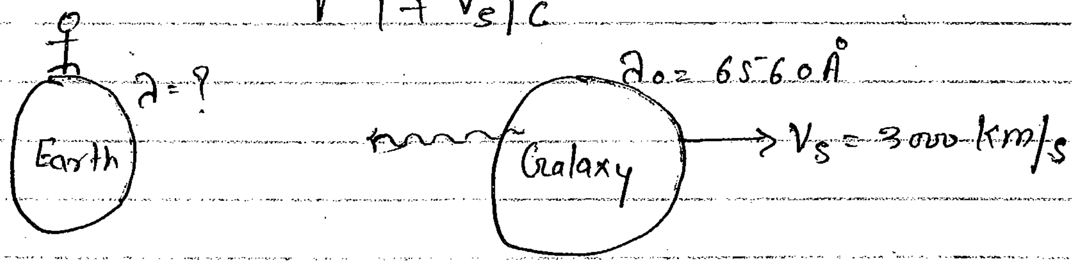
$\boxed{\nu = \nu_0}$ (No Transverse case in Doppler effect in sound.)

But In case of light.

$\boxed{\nu \neq \nu_0}$

59.

$$\nu = \nu_0 \sqrt{\frac{1 - v_s/c}{1 + v_s/c}} \qquad \nu < \nu_0$$



Here source is moving away from observer. So distance is increasing so freq. decreasing so wavelength is increasing. So

$$\nu = \nu_0 \sqrt{\frac{1 - v_s/c}{1 + v_s/c}}$$

$$\frac{c}{\lambda} = \frac{c}{\lambda_0} \sqrt{\frac{1 - v_s/c}{1 + v_s/c}}$$

$$\lambda = \lambda_0 \sqrt{\frac{1 + v_s/c}{1 - v_s/c}} = \lambda_0 \sqrt{\frac{1 + 10^{-2}}{1 - 10^{-2}}}$$

$$= 6560 \left[1 + 10^{-2}\right]^{1/2} \cdot \left(1 - 10^{-2}\right)^{-1/2}$$

$$= 6560 \left[1 + \frac{10^{-2}}{2}\right] \left[1 + \frac{10^{-2}}{2}\right]$$

$$\cancel{= 6560 \left[1 - \frac{10^{-4}}{4}\right]}$$

$$= 6560 \left[1 + 10^{-2} + \underbrace{\frac{1}{4} \times 10^{-4}}_{\text{neglect}}\right]$$

$$= 6560 + 65.60$$

$$\boxed{\lambda = 6625.60}$$

A-8

56

$V = v\hat{n} = v\hat{i}$ {becoz in L.T. direⁿ of velocity is x so in this direcⁿ unit vector is $\hat{i}$}

In S :-

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

In S' :-

$$\vec{r} = x'\hat{i} + y'\hat{j} + z'\hat{k}$$

Option (a) :-

$$\underbrace{\vec{r}'}_{\text{Vector}} = \gamma\,[\underbrace{(\hat{n}\cdot\vec{r})}_{\text{Scalar}}\ \underbrace{\hat{n}\cdot\vec{v}t}_{\text{Scalar}}]$$

So it can not be correct ans.

Option (b) :-

$$\vec{r}' = \gamma[(\hat{n}\cdot\vec{r})\hat{n} - \vec{v}t] + [\vec{r} - (\hat{n}\cdot\vec{r})\hat{n}]$$

$$(x'\hat{i} + y'\hat{j} + z'\hat{k}) = \gamma[x\hat{i} - v\hat{i}t] + [\cancel{x\hat{i}} + y\hat{j} + z\hat{k} - \cancel{x\hat{i}}]$$

Equating the cofficients of $\hat{i}, \hat{j}$ and $\hat{k}$ -

$$\boxed{x' = \gamma[x - vt]}$$

$$t' = \gamma[t - \vec{v}\cdot\vec{r}/c^2]$$

$1-\tanh^2\theta = \sec^2\theta \quad , \quad 1+\tan^2\theta = \sec^2\theta$

~~A.Q~~
2.38

$\frac{v}{c} = \tanh\theta \quad , \quad c=1$

$v = \tanh\theta$

$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\tanh^2\theta}}$

$= \frac{1}{\sqrt{\operatorname{sech}^2\theta}} = \frac{1}{\operatorname{sech}\theta} = \cosh\theta$

$\gamma = \cosh\theta$

L. T. :- t, x, y, z or x, y, z, t.

$t' = \gamma\left(t - \frac{xv}{c^2}\right) \rightarrow t' = \cosh\theta\left(t - \frac{x \cdot \sinh\theta}{\cosh\theta}\right)$

$t' = \cosh\theta\, t + (-\sinh\theta)\, x$

$x' = \gamma(x - vt) \rightarrow x' = \cosh\theta\, x - \sinh\theta\, t$

$= -\sinh\theta\, t + \cosh\theta\, x$

$y' = y \qquad y' = y$

$z' = z \qquad z' = z$

So

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{Transformation Matrix}} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Rotation about z-axis :-

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

* Relation b/w Force and Acceleration:-

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt} m\vec{v}$$

$$\boxed{\vec{F} = m_0 \frac{d}{dt}\left(\frac{\vec{V}}{\sqrt{1-V^2/c^2}}\right)} \qquad \because m = \frac{m_0}{\sqrt{1-V^2/c^2}}$$

$$\vec{F} = m_0 \left[\frac{\sqrt{1-V^2/c^2}\,\frac{d\vec{V}}{dt} - \vec{V}\frac{d}{dt}\sqrt{1-V^2/c^2}}{[1-V^2/c^2]}\right]$$

$$\frac{d}{dt}\sqrt{1-V^2/c^2} = \frac{1}{\cancel{2}\sqrt{1-\frac{V^2}{c^2}}}\left(-\frac{1}{c^2}\right)\frac{d}{dt}(V^2)$$

$$\frac{d}{dt}(\vec{V}^2) = \frac{d}{dt}(\vec{V}\cdot\vec{V}) = 2\vec{V}\left(\frac{d\vec{V}}{dt}\right)$$

$$\frac{d}{dt}\sqrt{1-\frac{V^2}{c^2}} = \frac{-\frac{1}{c^2}(\vec{V}\cdot\vec{a})}{\sqrt{1-V^2/c^2}}$$

$$\vec{F} = m_0 \frac{\left[\sqrt{1-V^2/c^2}\,\vec{a} + \frac{\vec{V}(\vec{V}\cdot\vec{a})}{\sqrt{1-V^2/c^2}}\right]}{(1-V^2/c^2)}$$

Case - I :- $\vec{V} \perp \vec{a}$

$$F = \frac{m_0}{\sqrt{1 - v^2/c^2}} \vec{a}$$

$$\boxed{\vec{F} = \gamma m_0 a} \quad \vec{a} \perp \vec{V}$$

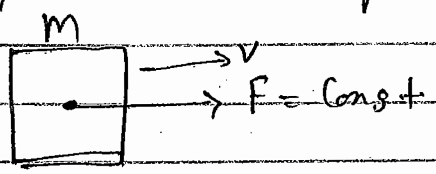
Case II :- $\vec{a} \parallel \vec{V}$

$$\boxed{\vec{F} = \gamma^3 m_0 \vec{a}} \quad \vec{a} \parallel \vec{V}$$

JEST

Q.34

State V/s Time Graph :-



Non Relativistic Case -

$$F = ma$$

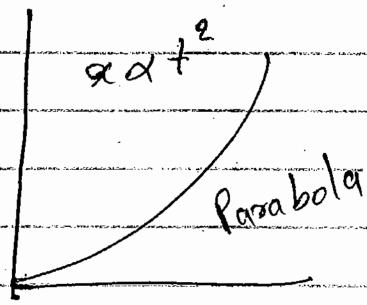
$$a = \frac{F}{m}$$

$$x = ut + \frac{1}{2} a t^2$$

$$x = 0 + \frac{1}{2} \frac{F}{m} t^2$$

$$\boxed{x = \frac{F}{2m} t^2}$$

equation of parabola



* ~~Lagrangian Formulation of Classical Mechanics~~ :-

Case - II Relativistic Case :-

$\vec{F} = \gamma^3 m_0 a \rightarrow a = \frac{F}{m_0 K}$ = Not Const.

not Const.

Constt.

$$\boxed{F = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \frac{dv}{dt}}$$

$$\int \frac{F}{m} dt = \int \frac{dv}{(1-v^2/c^2)^{3/2}} \longrightarrow$$ See Griffith E.M.t. Relativistic.

Hyperbola $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = K$

* **Density :-**

$$\text{Density} = \frac{\text{mass}}{\text{Volume}}$$

Rest :- $$\rho_0 = \frac{m_0}{V_0}$$

moving :- $$\rho = \frac{m}{V} = \frac{m_0\sqrt{1-v^2/c^2}}{V_0\sqrt{1-v^2/c^2}}$$

$$\boxed{\rho = \frac{\rho_0}{\left(1-\frac{v^2}{c^2}\right)}} \longrightarrow \text{mass density}$$

* **Charge Density :-**

$$\text{Density} = \frac{\text{Charge}}{\text{Volume}}$$

Rest $\Rightarrow$ $$\rho_0 = \frac{Q}{V_0}$$

moving $\Rightarrow$ $$\rho = \frac{Q}{V_0\sqrt{1-v^2/c^2}}$$

$$\boxed{\rho = \frac{\rho_0}{\sqrt{1-v^2/c^2}}} \longleftarrow \text{Charge density.}$$

Q.32

Sol^n K.E. = Total energy $- m_oc^2$

$$K = mc^2 - m_oc^2$$

$$= \frac{m_oc^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_oc^2 = m_oc^2\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{\lfloor 2} + \frac{n(n-1)(n-2)}{\lfloor 6}x^3 - -$$

$$K = m_oc^2\left[\left(\cancel{1} + \frac{1}{2}\frac{v^2}{c^2} - \frac{3}{8}\frac{v^4}{c^4} - - -\right) - \cancel{1}\right]$$

$$K = \frac{1}{2}m_o v^2 - \frac{3}{8}\frac{m_o v^2}{c^2} - - - -$$

first lowest order correction.

Q.29

Hamiltonian of a free particle :-

Non Relativistic Case :-

$$H = \frac{p^2}{2m} = K.E.$$

Relativistic Case :-

$$H = \sqrt{p^2c^2 + m_o^2c^4} = T.E.$$

$$H = \sqrt{p^2c^2 + m_0^2c^4}$$

$$= m_0c^2\sqrt{\frac{p^2}{m^2c^2} + 1}$$

$$= m_0c^2\left[1 + \frac{p^2}{m_0^2c^2}\right]^{1/2}$$

$$= m_0c^2\left[1 + \frac{p^2}{2m_0^2c^2} - \frac{1}{8}\frac{p^4}{m_0^4c^4} + \cdots\right]$$

$$\Rightarrow m_0c^2 + \frac{p^2}{2m_0} - \frac{p^4}{8m_0^3c^2} \leftarrow \text{first correction}$$

non Relativistic.

So option ⓐ is correct.

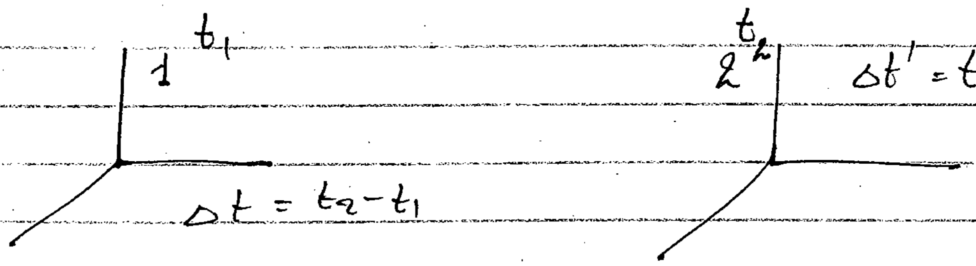
Lagrangian $\Rightarrow \boxed{L = -m_0c^2\sqrt{1 - v^2/c^2}}$

Q.41 The ordering of two events in absolute (does not change) if the invariant interval is?

(a) Space-like (b) time-like

(c) light-like (d) either space-like or time-like.

Let two events occurs -



① If $\Delta t > 0$ in S & $\Delta t' > 0$ in S' then ordering of event not changing.

② If $\Delta t > 0$ & $\Delta t' < 0$ then ordering changing.

$$\Delta t' = \frac{\Delta t - \frac{\Delta x v}{c^2}}{\sqrt{1 - v^2/c^2}}$$

If $\Delta t > 0$ in S and $\Delta t' > 0$ in S' } ordering of event not changing.

$\Delta t > 0$, $\Delta t' < 0$ ordering of event changing.

$$\Delta t' = \frac{\Delta t \left[1 - \frac{\Delta x}{\Delta t}\frac{v}{c^2}\right]}{\sqrt{1 - v^2/c^2}}$$

$$\Delta t' = \Delta t \frac{\left[1 - \frac{\Delta x}{c\Delta t}\frac{v}{c}\right]}{\sqrt{1 - v^2/c^2}}$$ { ∵ here $\frac{v}{c} < 1$ ∵ $v < c$ }

$\frac{\Delta x}{c\Delta t} = 1$ ← Light like

$\frac{\Delta x}{c\Delta t} > 1$ ← Space like

$\frac{\Delta x}{c\Delta t} < 1$ ← time like

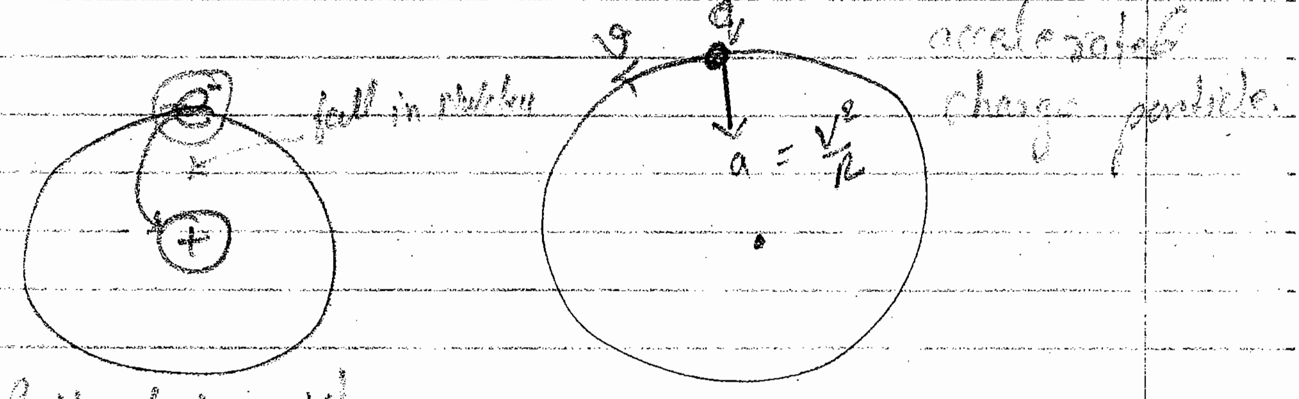
← ordering does not change

ordering is change.

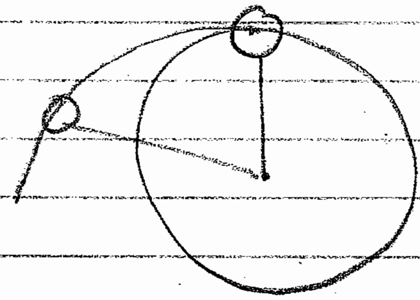
option (C) ← (B)

Q.(96) → (b)

Q.(97)



Rutherford model



Ans (C)

Q.53 Lorentz boost = Lorentz's transformation.

$u' = ct' + x'$

$= c\left(t - \frac{xv}{c^2}\right)\gamma + (x - vt)\gamma$

(b)

(S1)

$\tan\theta = \frac{\tan\theta'}{\sqrt{1-\frac{v^2}{c^2}}}$ ← angle in Rest frame.

(S.1)

So $\tan\theta' = \frac{\tan\theta}{\sqrt{1-\frac{v^2}{c^2}}}$ s

$50°$ ← Rest angle

$\therefore \sqrt{1-\frac{v^2}{c^2}} < 1$

So $\tan\theta' = \gamma\tan\theta$

$\tan\theta' > \tan\theta$

$\boxed{\theta' > \theta}$

(33) $\ominus \xrightarrow{e^-}$ $\quad \xleftarrow{e^+} \bigcirc$

Photon ∿∿ $e^- e^+$ ∿∿ Photon

anihilation

(by taking in contact).

$$T.K._i = T.E._f$$

$$m_0c^2 + m_0c^2 = h\nu + h\nu$$

$$2m_0c^2 = 2\frac{hc}{\lambda}$$

2×0.5 M.eV $= \frac{2}{\lambda} \times 1242$ eV - nanometer

1×10^6 $\cancel{eV}$ $= \frac{2}{\lambda}\times 1242$ $\cancel{eV}$ nanometer

$$10^6 = \frac{2}{\lambda}\times 1242 \text{ n.m.}$$

$$\lambda = 1242\times 2\times 10^{-6} \text{ n.m}$$

$= 2.484\times10^{-12}$ pikometer

(18) $\Delta t' = \dfrac{\Delta t - \Delta x v}{\sqrt{1-v^2/c^2}}$ $\quad v \rightarrow$ relative velo.

* Lagrangian Formulation of Classical Mechanics:-

Degree of freedom :- {DOF}:- It is the number of independent coordinates required to describe dynamics of system. It is represented by 'f'.

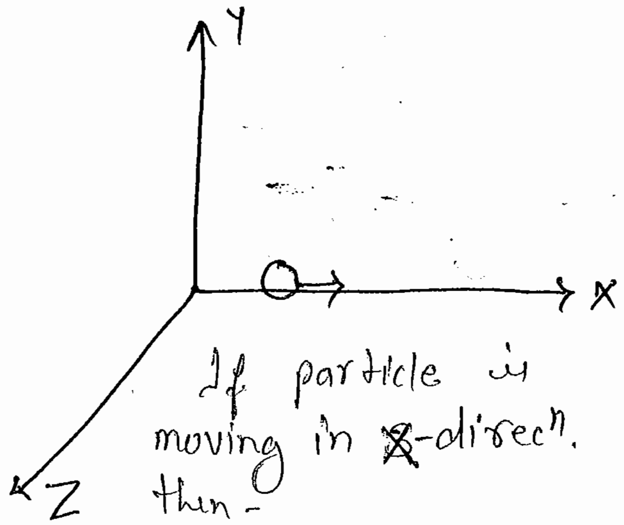
for N-particles moving freely in 'd' dimensional world.

$$\boxed{f = Nd}$$

N = No. of particle
d = ~~dimensioned~~ dimension of particle.

If there is some constraints then -

$$\boxed{f = Nd - k}$$ where k = No. of constraints.



If particle is moving in ~~x~~-direc^n. then -

$d = 3,\ N = 1,\ k = 2$ (means it can't travel in y and z)

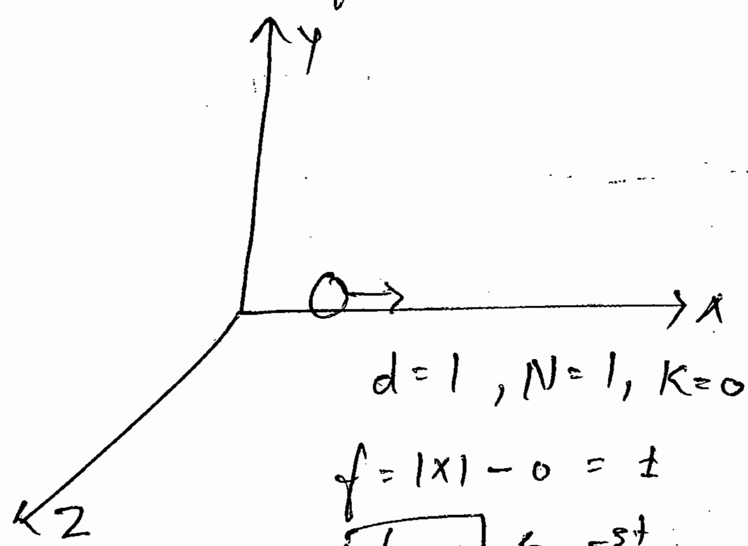
So $f = 1 \times 3 - 2$

$= 3 - 2$

$$\boxed{f = 1}$$

$\begin{Bmatrix} y = 0 \\ z = 0 \end{Bmatrix}$

IInd way of description.



$d = 1,\ N = 1,\ k = 0$

$f = 1 \times 1 - 0 = 1$

$$\boxed{f = 1}$$

Ist way of description

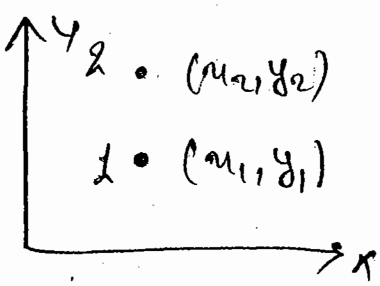
If 2 particle is moving freely in xy - plane.

$N = 2,\ d = 2$
$k = 0$

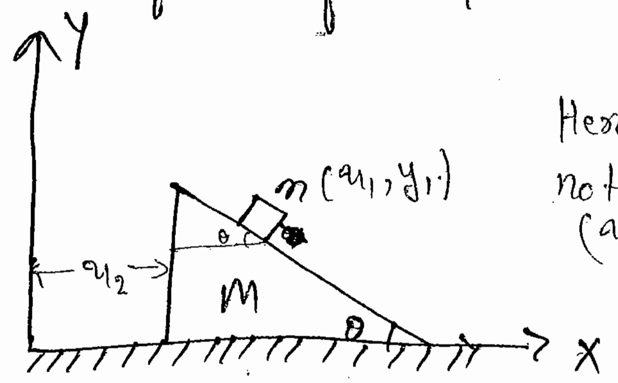
$f = Nd - k$

$f = 2 \times 2 - 0$

$$\boxed{f = 4}$$



Q. Block and Wedge move in vertical plane. what is D.O.f. of system is ?



Here (x_1, y_1) and x_2 are not independent. becoz Here (x_1, y_1) are dependent on each other

Sol^n

$f = Nd - K$

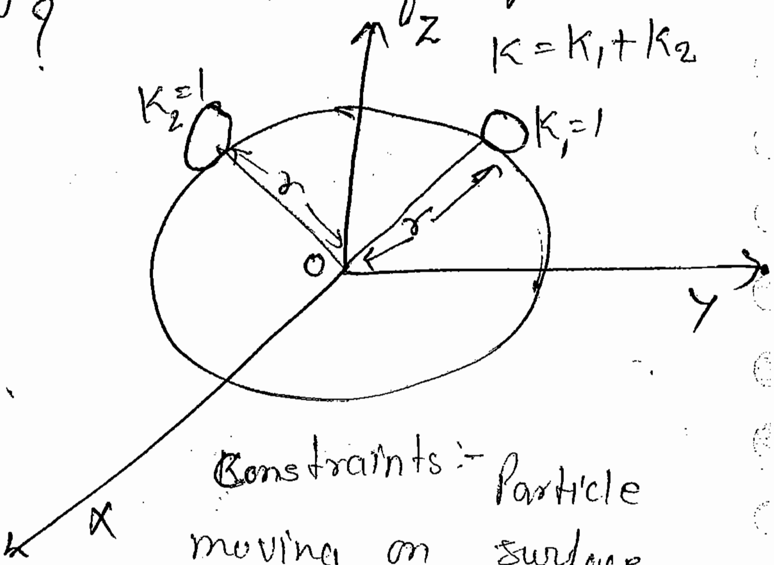
$N = 2, \; d = 2, \; K = 2$

$f = 2 \times 2 - 2$

$\boxed{f = 2}$

Q. Two particles are moving on surface of sphere find degree of freedom ?

$K = K_1 + K_2$



Sol^n

$f = Nd - K$

$N = 2, \; d = 3, \; K = 2$

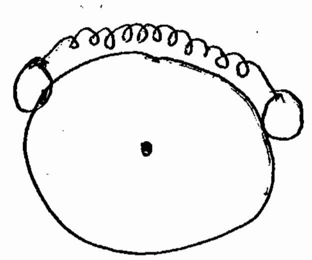
$f = 2 \times 3 - 2$

$= 6 - 2$

$\boxed{f = 4}$

Constraints :- Particle moving on surface any where but distance from the center is always r.

* Particle Connected by spring :-

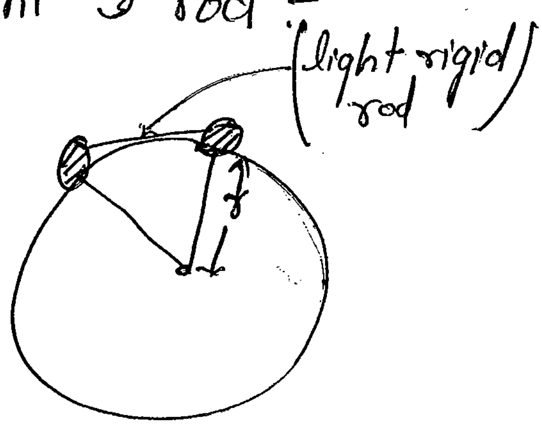


Spring does not put any condition (Constraints).

So. $\boxed{f = 4}$

* Particle Connected by light rod :-

Here there is extra condition (Constraints) arise that distance between particle is fixed.



So $K = 3$

$f = Nd - K$

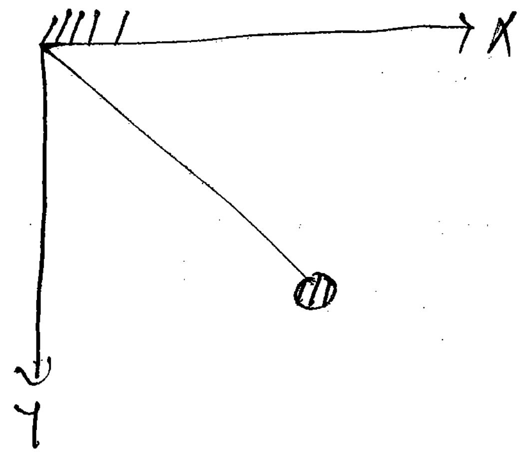
$= 2 \times 3 - 3$

$= 6 - 3$

$\boxed{f = 3}$

* Simple Pendulum :-

In simple pendulum, motion is confined in one plane.

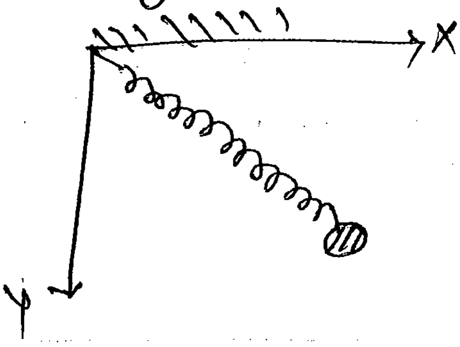


$N = 1$	$N = 1$
$d = 2$	$d = 3$
$K = 1$	$K = 2$
$f = Nd - K$	$f = Nd - k$
$= 2 \times 1 - 1$	$= 1 \times 3 - 2$
	$= 3 - 2$
$\boxed{f = 1}$	$\boxed{f = 1}$

① Confined in one plane

② Distance from point of Suspension is l.

① If string is replaced by spring (or flexible string) :-



$d = 2$

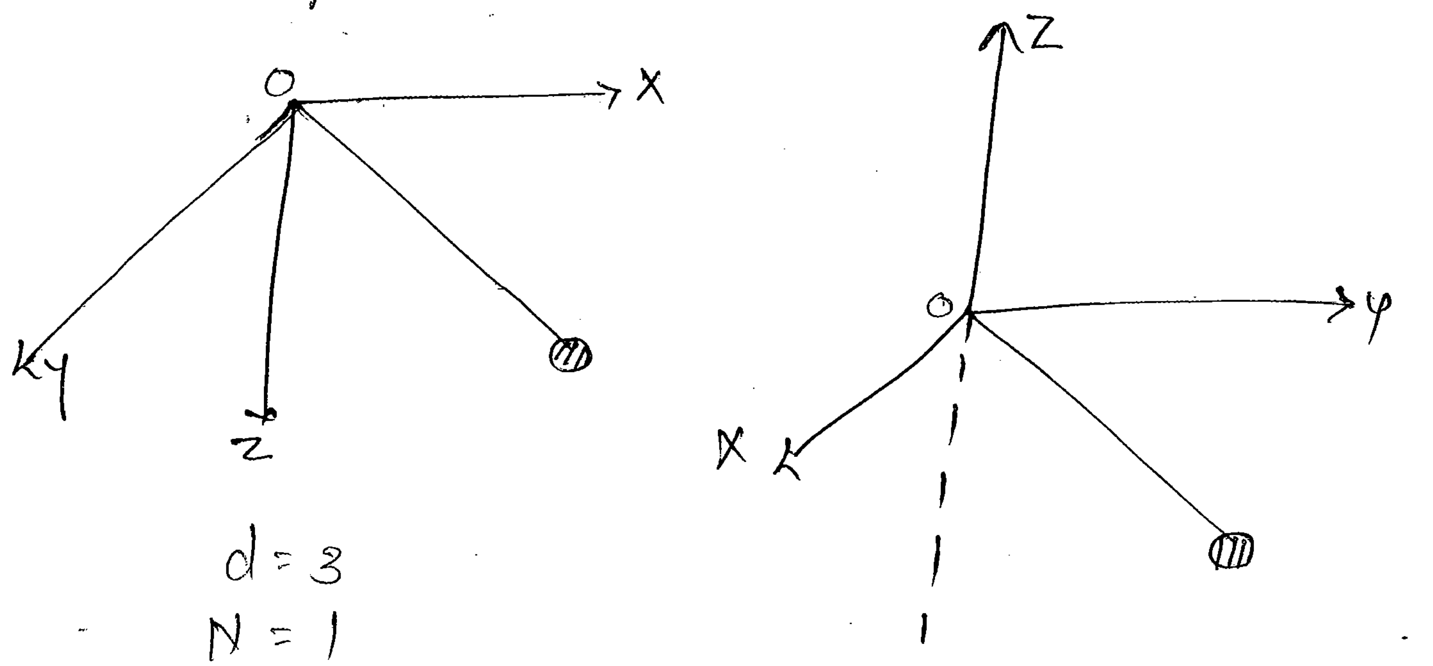
$K = 0$

$N = 1$

$\boxed{D.O.f. = 2}$

* Spherical Pendulum :-

Motion is not confined in one plane.



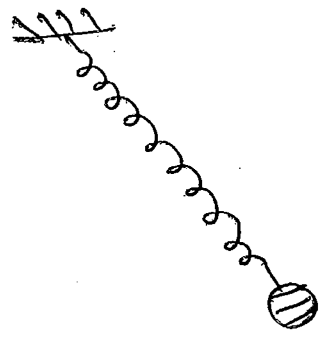
$d = 3$

$N = 1$

$K = 1$ (distance from O is fixed.)

$\boxed{D.O.F. = 2}$

If string is replaced by flexible string or Spring :-



$d = 3$
$K = 0$
$N = 1$

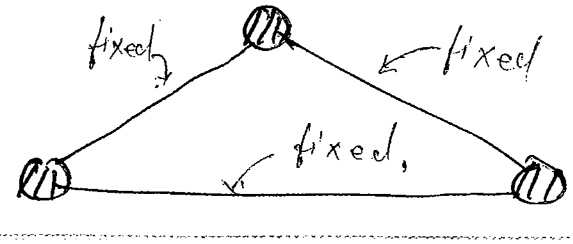
$\boxed{DOF = 3}$

Q. Three particle are connected by a light rod to each other as shown in figure. If system is moving in 3-Dimension then what is it's degree of freedom.

Sol^n $K = 3$, $N = 3$, $d = 3$

$f = Nd - K = 3 \times 3 - 3$

$\boxed{f = 6}$



If moving in 2-D.

$d = 2$ $DoF = 3 \times 2 - 3$

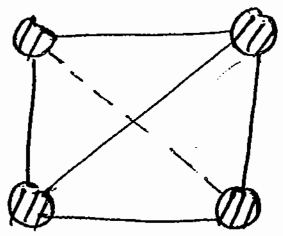
$N = 3$ $= 6 - 3$

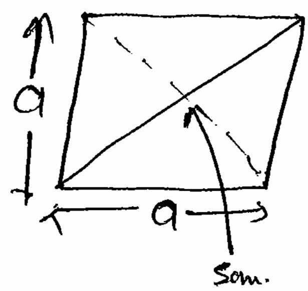
$K = 3$ $\boxed{DoF = 3}$

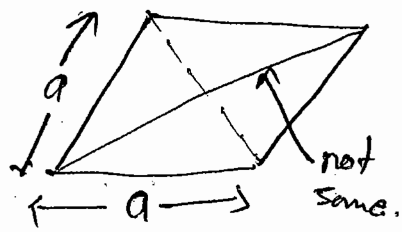
Q. 10 particles are connected to each other by light rod and system is moving in 3-D what is degree of freedom.

Sol^n If $N \geq 3$ and distance b/w the particle is fixed. And the system is moving in 3-D then degree of freedom is equal to 6.
If it is in 2-D it is equal to 3.
If it is in 1-D it is equal to 1.
True for any rigid body

for $N = 4$:







$DOF = Nd - K$

$= 4 \times 3 - 6$

$= 12 - 6$

$\boxed{DOF. = 6}$

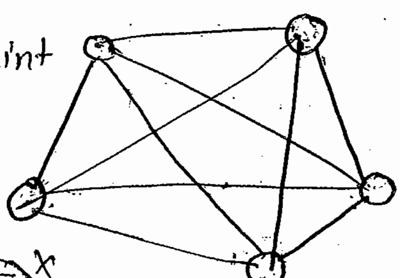
for $N = 5$:

One constraint is redundant (not require).

$N = 5$

$K = 10$

$d = 3$



$DoF = 5 \times 3 - 10 = 15 - 10 = 5$ not true becoz by statemen it is $= 6$

* Degree of freedom of rigid Body :-

$$\boxed{f = \frac{d(d+1)}{2}}$$

d = dimentionality of World.

3-D → $f = 6$ = 3 rotation + 3 translation
2-D → $f = 3$ = 2 translation + 1 rotation
1-D → $f = 1$ = 1 translation.

---

* Constraints (Conditions)

or

Geometrical Condition on Coordinates

{There is a relation between Coordinates}

* Types of Constraints :-

Constraints

Holonomic (Complete)

⇒ Relation b/w Coordinates is algebric equation

e.g. ⇒ $x^2 + y^2 = l^2$

⇒ If relation b/w coordinates is differential equation then it must be reducible to algebraic equation.

e.g. $x\,dy + y\,dx = 0$

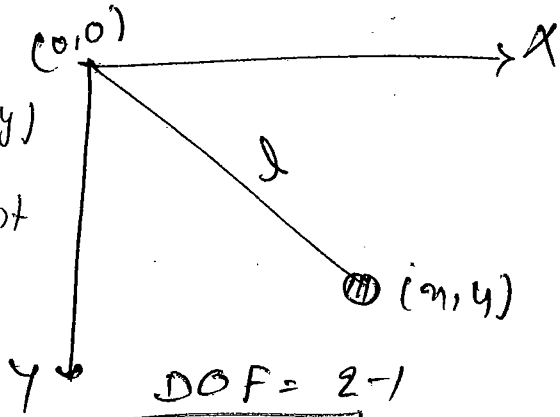
$\int d(xy) = 0 \Rightarrow xy = \text{Constt.}$ ← Algebraic eq^n.

Non-Holonomic

If relation b/w coordinates is differential equation

* Holonomic Constraints :-

1. Simple Pendulum :-



Variable Coordinates $= 2 \Rightarrow (x, y)$

$x^2 + y^2 = l^2 \longrightarrow$ Constraint equation. (Holonomic)

$\boxed{y^2 = l^2 - x^2}$

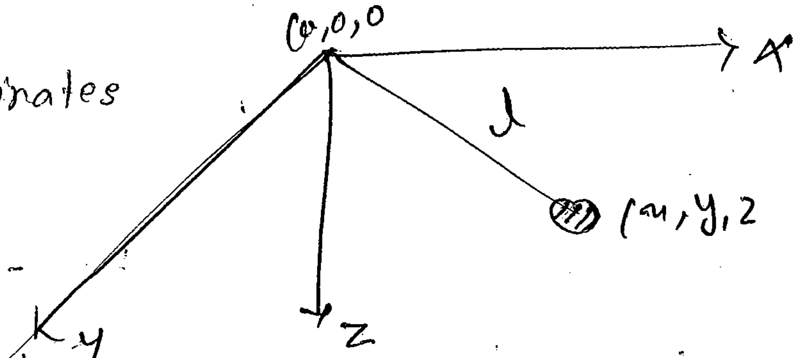
So if we know 'x' we can find 'y'. So it is dependent.

DOF = 2-1

$\boxed{\text{DOF} = 1}$

$\boxed{\text{DOF} = \text{No. of Variable Coordinate} - \text{No. of Constraint equation (Holonomic)}}$

2. Spherical Pendulum :-



No. of variable Co-ordinates $= 3$ (x, y, z)

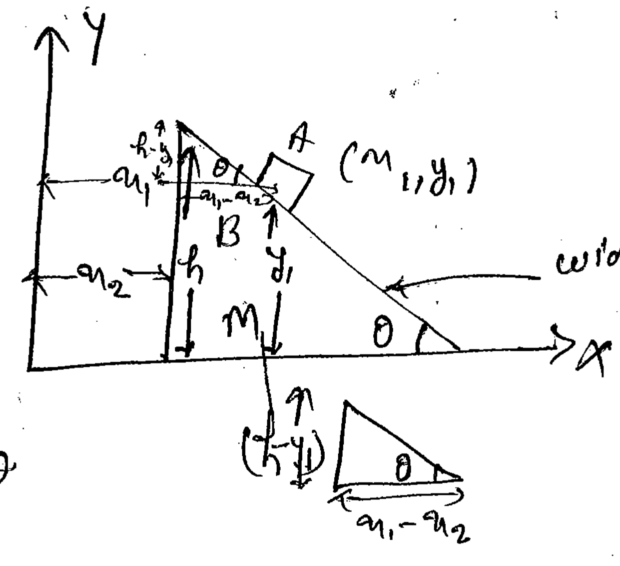
Constraints equation -

$x^2 + y^2 + z^2 = l^2$ (Holonomic)

$\boxed{\text{DOF} = 3 - 1 = 2}$

3. Block & Wedge :-

Here h and θ are fixed.



$\tan\theta = \dfrac{h - y_1}{x_1 - x_2}$

$h - y_1 = (x_1 - x_2)\tan\theta$

$$\boxed{y_1 = h - (x_1 - x_2)\tan\theta}$$ ← Constraint equation

DOF = 3 − 1

$\boxed{DOF = 2}$

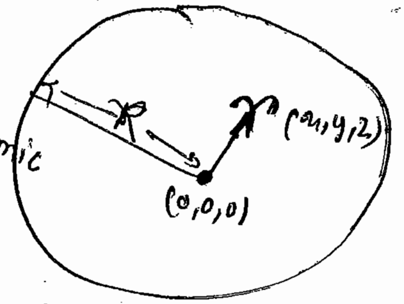
Non Holonomic Constraintes

Constraint equations are either inequality or non integrable differential equations.

* It does not reduce degree of freedom.

Ex- A particle (fly) moving ~~inside~~ inside a sphere.

$$\boxed{\sqrt{x^2+y^2+z^2} \le R}$$ ← Non Holonomic



$d = 3$
$N = 1$
$K = 0$

$DOF = Nd - K$
$= 1\times 3 - 0$

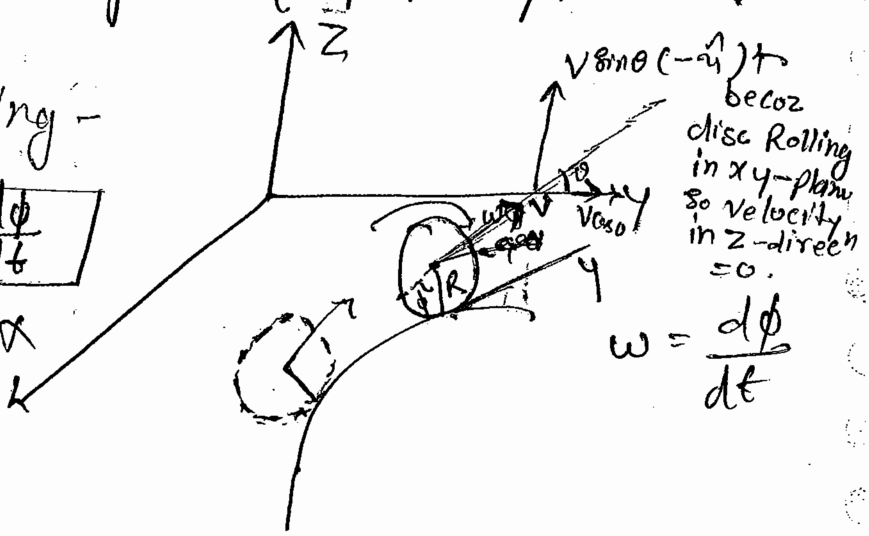
$\boxed{DOF = 3}$

Here there is one constraint ⇒ the fly can not go out of the sphere. but it is non-Holonomic.

Ex- Disk Rolling on a plane (XY-Plane):-

Condition ~~of~~ for Rolling -

$\boxed{V = R\omega}$ $\boxed{V = R\dfrac{d\phi}{dt}}$



$\omega = \dfrac{d\phi}{dt}$

$V_y = V\cos\theta$

$\dfrac{dy}{dt} = R\dfrac{d\phi}{dt}\cos\theta$

$$\boxed{dy = R\,d\phi \cos\theta} \quad \text{(i)}$$

$$V_x = -V\sin\theta$$

$$\boxed{dx = -R\,d\phi \sin\theta} \quad \text{(ii)}$$

Divide (i) by (ii)

$$\frac{dy}{dx} = -\cot\theta$$

$$\boxed{\int dy = -\int \cot\theta \, dx}$$

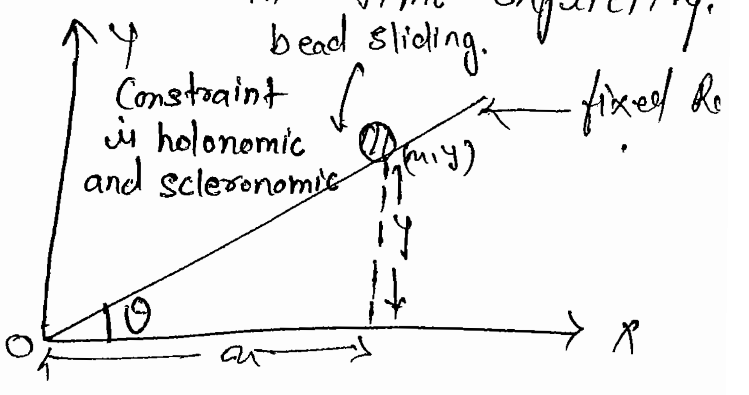
"Differential equation for x and y θ is variable So it can not be integrated.
∴ Rolling is a non-holonomic constraint. When motion is along a curved line. and it is holonomic constraint, if motion is along a straight line."

* Holonomic and Non-Holonomic Constraints have two types.

1. Scleronomic (rigid) : Time Independent:-

Therefore constraint equation do not contain time explicitly.

Example :- Bead sliding on a fixed rod.



Constraint equation -

$$\tan\theta = \frac{y}{x}$$

$\boxed{y = x\tan\theta}$ → does not contain time.

Dof = No. of variable − (No of Constraint eqn)

= 2 − 1

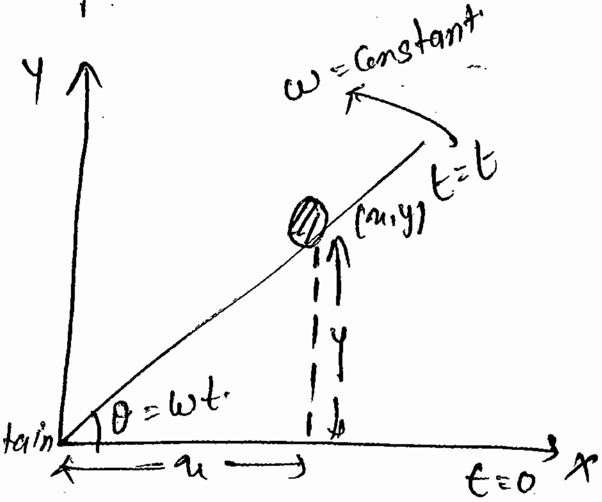
$\boxed{Dof = 1}$

2. Rheonomic Constraint: (Non Rigid)

{Time Dependent}

Constraint equation explicitly contain time.

Ex:- Bead sliding on a rotating rod.



Equation of Constraint -

$$\tan\theta = \frac{y}{x}$$

$$\boxed{y = x \tan\omega t} \Rightarrow \text{It contain time.}$$

⇓

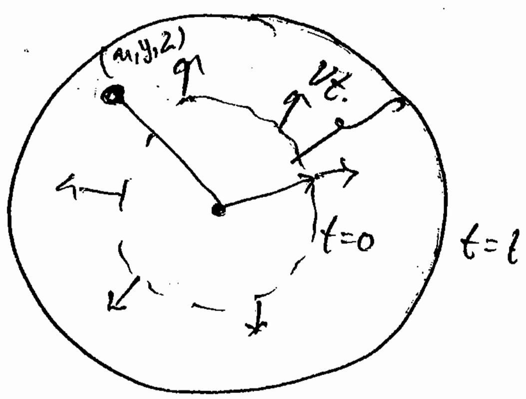
Holonomic and Rheonomic.

Ex:- Particle moving inside a expanding sphere.

$$\boxed{\sqrt{x^2+y^2+z^2} \le R_0 + vt}$$

⇓

Non-Holonomic and Rhonomic.



* Generalised Coordinates : Convenient Coordinates

Note :- Do not use concepts about dynamics of System lernt from Newtonian Mechanics.

* "Convenient Coordinates chosen to simplify problem are called generalised coordinates."

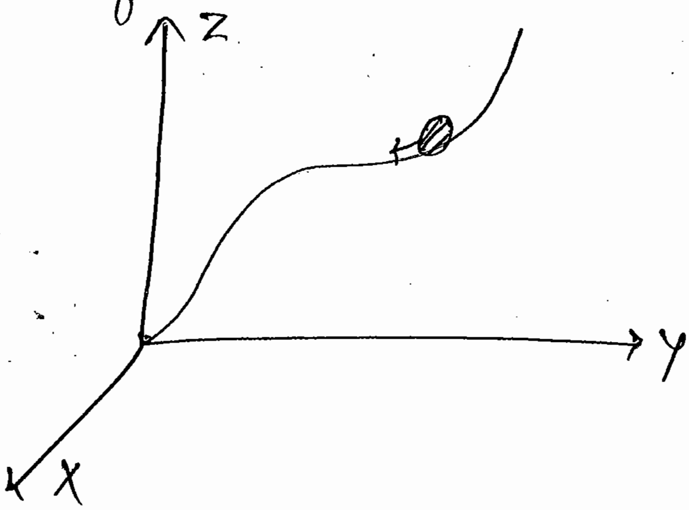
"In most cases number of generalised coordinates is equal to Degree of freedom."

2. A particle moving along Curve line (fixed) what is degree of freedom of particle?

Sol^n

$N = 1$
$k = 1$
$d = 3$

$f = 3 \times 1 - 1 = 2$ ×

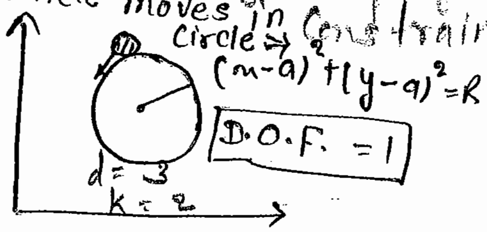


but it is not true
Here degree of freedom

$\boxed{f = 1}$

because A line is formed by intersection of two planes.
So ~~degree of freedom~~ of one plane = 1
and for another plane = 1
So total Constraints = 1+1 = 2
$f = 3 \times 1 - 2 = 1$ ✓

If particle moves in Constrained Circle ⇒ $(x-a)^2 + (y-a)^2 = R$



Note :- If particle moves along a fixed line (it may be a curved line) or straight line then $\boxed{D.O.F = 1}$

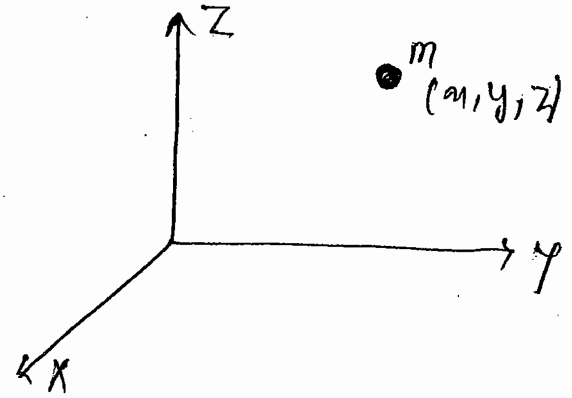
Because a line is formed by intersection of two planes.

* Kinetic Energy in Different Coordinate System :-

(1) Cartesian Co-ordinate :-

$$K.E. = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

(Most Useful)



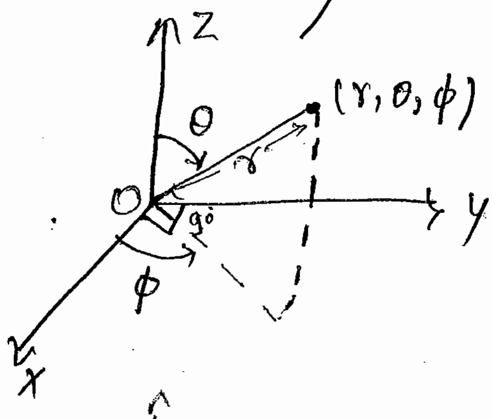
(2) Spherical Polar :-

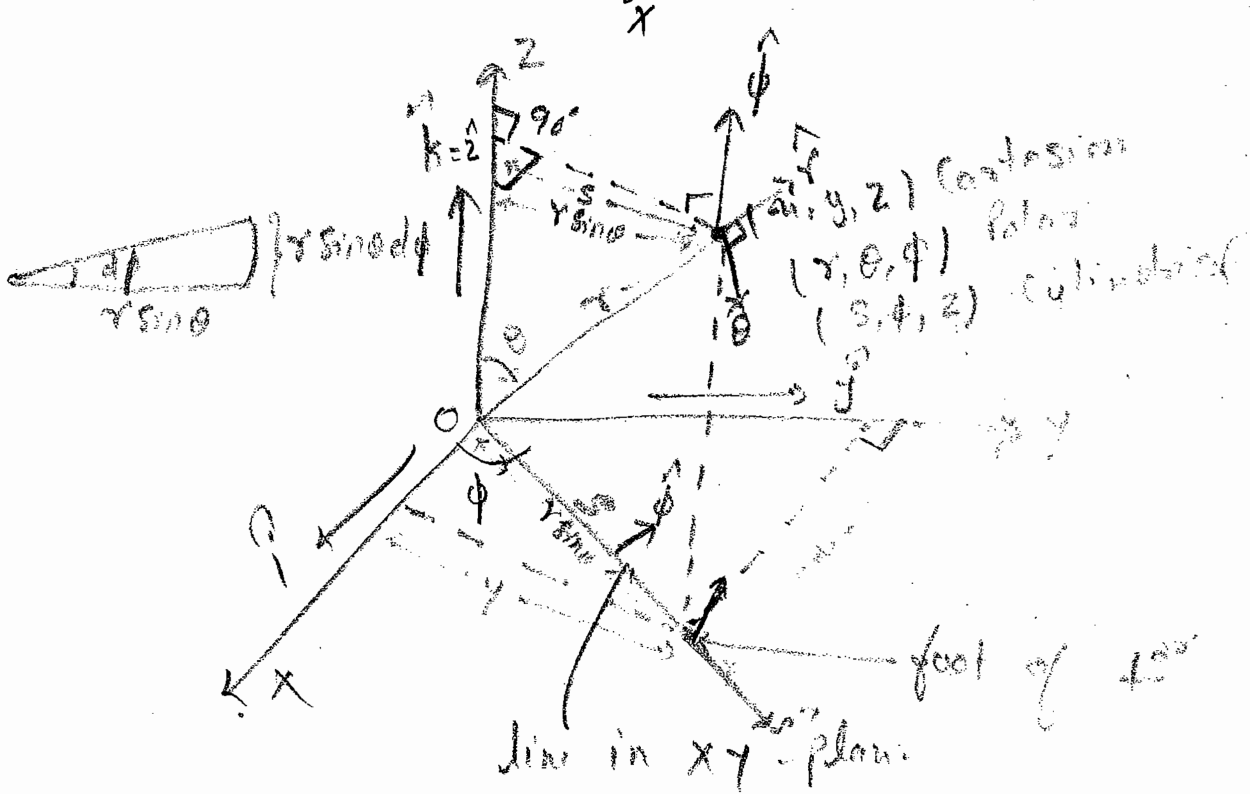
$$K.E. = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right)$$

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$





* How Write K.E. in Different Coordinate System:

Small displacement of particle in different coordinates.

$\Rightarrow$ $\boxed{d\vec{s} = dx\hat{i} + dy\hat{j} + dz\hat{k}}$ In Cartesian Coordinate.

$\Rightarrow$ In Spherical Polar :-

$$\boxed{d\vec{s} = dr\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}}$$

$\Rightarrow$ In Cylindrical Coordinate :-

$$\boxed{d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}}$$

* Velocity :-

$$\vec{v} = \frac{d\vec{s}}{dt} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

$$\vec{v} = \frac{d\vec{s}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\hat{\phi}$$

$$v = \frac{d\vec{l}}{dt} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$

* Kinetic Energy :-

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$K = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin\theta\,\dot{\phi}^2)$$

$$K = \frac{1}{2}m(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2)$$

Note: If a particle is moving in a plane/space and force on it is always directed towards a point or particle is attached to a point by a string/spring or rod then use of plane polar/spherical polar is convenient.

⇒ If case is not so in a plane then we can use Cartesion.

⇒ Plane Polar :-

If we remove z from cylindrical coordinate system then we get plane polar.

* How to write K.E. for more than one particle cases :-

1. fist Method :- More General Method.

$$K.E. = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2)$$

2. Second Method :-

Use superposition method to calculate speed of second particle if speed of first particle is given. (less General).

[Use mostly for motion in one plane]

* Generalised Coordinate :-

Independent Coordinates (Varia.. chosen to simplify the problem.

Notation :- q_i

$$\frac{\partial q_i}{\partial q_j} = \delta_{ij}$$

$$\boxed{\frac{\partial q_1}{\partial q_2} = 0}$$

Generalised Velocity :-

$$\boxed{\dot{q}_i = \frac{dq_i}{dt}}$$

⇒ In lagrangean formulation q_i and $\dot{q}_i$ are taken to be independent while solving the problem.

$$\boxed{\begin{aligned} \frac{\partial \dot{q}_i}{\partial q_i} &= 0 \\ \frac{\partial q_i}{\partial \dot{q}_i} &= 0 \end{aligned}}$$

← Before problem has been solved

$$\boxed{\frac{\partial \dot{q}_i}{\partial \dot{q}_j} = \delta_{ij}}$$

After problem is solved q_i and $\dot{q}_i$ may turn out to be dependent (actually).

* K.E. of Double Pendulum :- {both are moving in same plane}

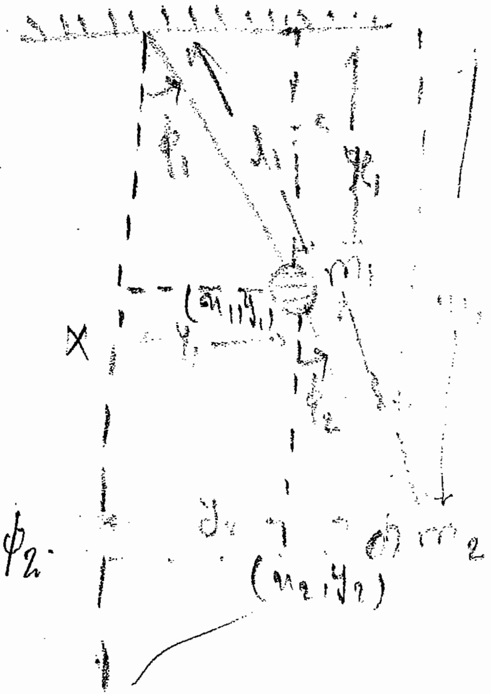
Degree of freedom of system:-

$N = 2, \; d = 2, \; K = 2$

$f = Nd - K$

$f = 2 \times 2 - 2 = 4 - 2$

$\boxed{f = 2}$



Generalised Co-ordinates are ϕ_1 and ϕ_2.

$x_1 = l_1 \cos\phi_1 \; , \; \dot{x}_1 = -l_1 \sin\phi_1 \dot{\phi}_1$

$y_1 = l_1 \sin\phi_1 , \; \dot{y}_1 = l_1 \cos\phi_1 \dot{\phi}_1$

$x_2 = l_1 \cos\phi_1 + l_2 \cos\phi_2 \; , \; \dot{x}_2 = -\left[l_1 \sin\phi_1 \dot{\phi}_1 + l_2 \sin\phi_2 \dot{\phi}_2\right]$

$y_2 = l_1 \sin\phi_1 + l_2 \sin\phi_2 \; , \; \dot{y}_2 = \left[l_1 \cos\phi_1 \dot{\phi}_1 + l_2 \cos\phi_2 \dot{\phi}_2\right]$

We write K.E. of the system by **first method** -

$$K.E. = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

So $$K.E. = \frac{1}{2} m_1 [l_1^2 \dot{\phi}_1^2] + \frac{1}{2} m_2 [l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)]$$

So $$\boxed{T = \frac{1}{2} \left[m_1 l_1^2 \dot{\phi}_1^2 + m_2 \left(l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)\right)\right]}$$

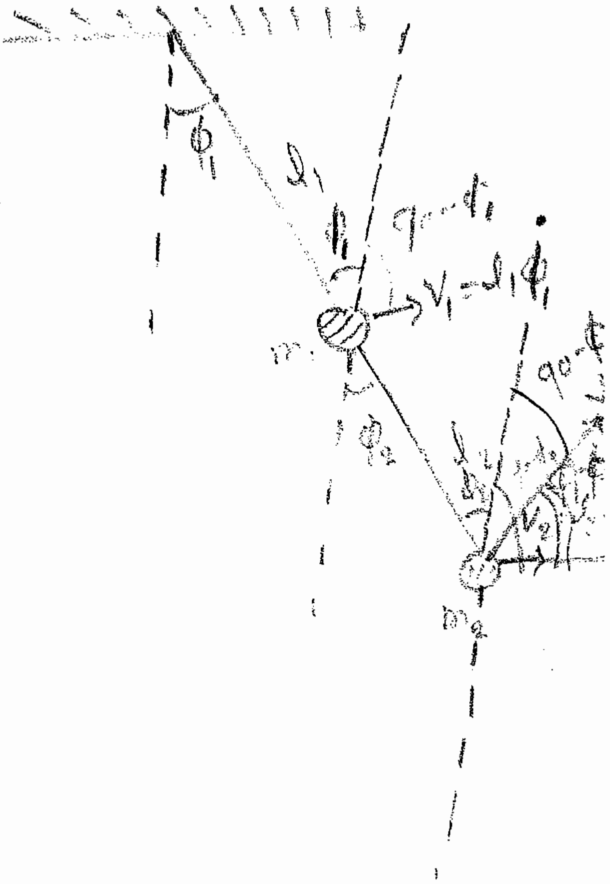
* Second Method :-

Velocity Superposition Method :-

$$\boxed{\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi}}$$

$$\vec{v}_1 = 0 + l_1\dot{\phi}_1\hat{\phi}_1$$

$$\boxed{\vec{V}_1 = l_1\dot{\phi}_1\hat{\phi}_1} \quad \boxed{\vec{V}_2 = l_2\dot{\phi}_2\hat{\phi}_2}$$



Motion of m_2 depends upon motion of m_1.

So K.E. of $m_1 = \frac{1}{2}m_1 v_1^2$

$$= \frac{1}{2}m_1 l_1^2\dot{\phi}_1^2$$

Net velocity of $m_2 = \sqrt{v_1^2 + v_2^2 + 2v_1v_2\cos(\phi_1 - \phi_2)}$

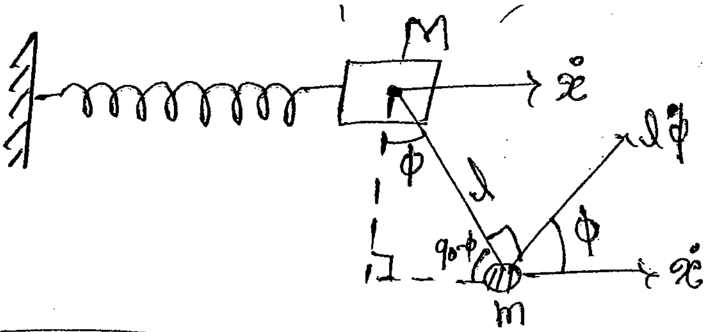
So K.E. of $m_2 = \frac{1}{2}m_2(\text{net velocity})^2$

$$\text{K.E. of } m_2 = \frac{1}{2}m_2\left[l_1^2\dot{\phi}_1^2 + l_2\dot{\phi}_2^2 + 2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)\right]$$

So Total K.E. of the system is -

$$\boxed{T = \frac{1}{2}\left[m_1l_1^2\dot{\phi}_1^2 + m_2(l_1^2\dot{\phi}_1^2 + l_2\dot{\phi}_2^2 + 2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)\right]}$$

Q.



What is K.E. of the system?

Sol^n

$\boxed{DOF = 2}$

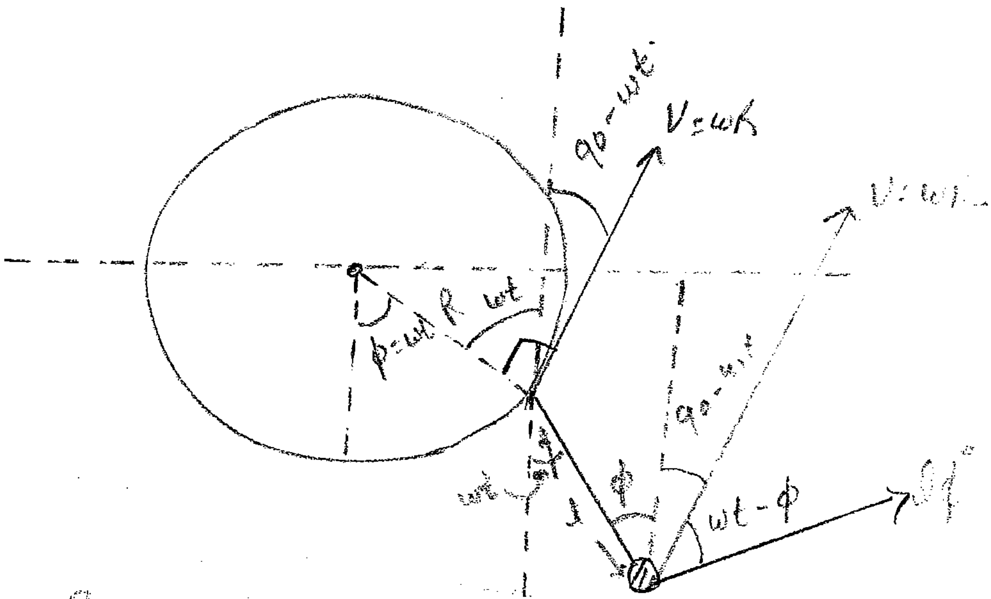
$V_1 = \dot{x}$

$V_2 = l\dot{\phi}$

$$\boxed{K.E. = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m\left(\dot{x}^2 + l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi\right)}$$

A-10

Q.20



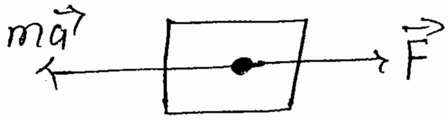
$$\boxed{K.E. = \frac{1}{2} m\left(l^2\dot{\phi}^2 + \omega^2 R^2 + 2l\dot{\phi}\omega R\cos(\omega t - \phi)\right)}$$

* Lagrangian Formulation :-

It is baised on following two approaches -

1. D'Alembert's Principle. +
2. Principle of virtual work. [Baised on Physics]

D'Alembert's principle converts Dynamic system into static system. by assuming a revese force.



$$\underline{\underline{\vec{F} - m\vec{a} = 0}}$$

Principle of Virtual Work:-

$$\sum_{i=1}^{3N} \vec{f_i} \cdot \delta\vec{r_i} = 0$$

2. Variation Calculus (principle) Approach :- [Baised on Mathematics]
   (Derivation of Schrodinger eq^n).

$$\text{Action } (S) = \int_{t_1}^{t_2} L\,dt$$

dynamics of system is such that action is extremum (only that dynamics is allowed in which action is extrimum).

$\delta S = 0$ [Condition for extremum]

$\delta \int L\,dt = 0$ [Condition for extremum]

↓

This gives as Lagrangian's Equation

* Lagranges' Equation :-

First form : $\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i}$

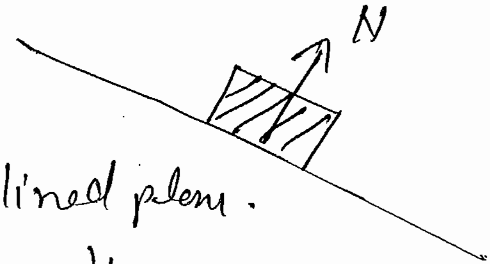
$T = K.E.$

$Q =$ Generalised force (It includes all forces except constraint forces)

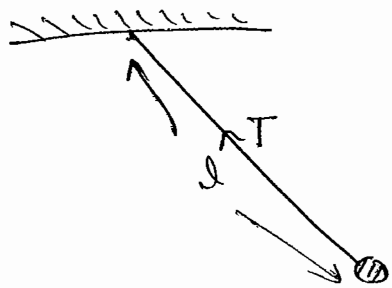
* Constraint forces :-

forces arising due to some constraints.

Example :- Normal reation, tension etc.



Normal reaction is arised when object is started to slide down on a Inclined plan.
It is due to contact between the inclined plane surface and the object ~~betwee~~ is produced the normal reaction.



* Second form:-

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i}$$

Q_i → Generalised force (It includes those forces which potential energy can not be written.).

$L = T - V$ = Lagrangian

V → Potential Energy

* For Constructive system (monogenic system):-

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$$

(Monogenic system is that system for which potential energy can not be written.)

* Relation between Q_i and f_i (actual force's Component)

Q_i is may or may not be component of actual force.

$$\boxed{Q_i = \sum_i f_j \frac{\partial r_i}{\partial q_i}}$$ ⇒ Used to find Q_i if f_i is given.

$$\boxed{Q_\theta = \sum F_j \frac{\partial r_j}{\partial \theta}}$$

Imp.

* Relation b/w Generalised force and Generalised potential :- $\{U(q_i, \dot{q}_i)\}$:-

$$\boxed{Q_i = -\frac{\partial U}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_i}\right)}$$

Generalised force may or may not be unit of actual force.

Same as:-

$\vec{F} = -\vec{\nabla} V(x, y, z)$

$\vec{F}_x = -\frac{\partial V(x)}{\partial x}$

* Correspondence b/w Q_i and q_i :-

"Product Q_i, q_i always has dimension of work. Although q_i and Q_i may not have dimension of length and force."

q_i	Q_i
(actual cordinate) x	F_x (actual component of force)
(Angle) θ	F_θ (torque)
(Thermodynamic co-ordinate variable) Volume	P (Pressure)
temp.	entropy

* Generalised Momentum :-

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i}$$

Q_i → generalised force.

dimension

dimension

$$\boxed{p_i = \frac{\partial L}{\partial \dot{q}_i}}$$

$$\vec{F} = \frac{d(\vec{p})}{dt}$$

It may or may not depend on q_i and $\dot{q}_i$

$\frac{\partial p_i}{\partial q_i}$ = ~~0 may~~ may or may not be equal to zero.

$\frac{\partial p_i}{\partial \dot{q}_i} = 0$ or may not be equal to zero.

p_i may or may not be component of actual momentum.

However product p_i and q_i always dimension of angular momentum.

	q_i	p_i
(actual Co-ordinate)	x	p_x (actual momentum)
(angle)	θ (dimension less)	p_θ (angular momentum)

CSIR
2013 Dec

Q. $V = x^2+y^2+\frac{z^2}{2}$ which compount of angular momentu is conserved.?

Sol^n

$$L = \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\,\dot{\phi}^2) - \left[x^2+y^2+\frac{z^2}{2}\right]$$

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$$L = \frac{1}{2}m\{\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\,\dot{\phi}^2] - \left[r^2\sin^2\theta\cos^2\phi + r^2\sin^2\phi\sin^2\theta + \frac{r^2\cos^2\theta}{2}\right]$$

$$= \frac{1}{2}m[\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\,\dot{\phi}^2] - \left[r^2\sin^2\theta + r^2\frac{\cos^2\theta}{2}\right]$$

eq^n of r:- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r}$

$$m\ddot{r} - [mr\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2 - 2r\sin^2\theta + r\cos^2\theta]$$

eq^n of θ:-

ϕ = Cyclic

p_ϕ = Conserved

$$\boxed{L_z = \text{Constant}}$$

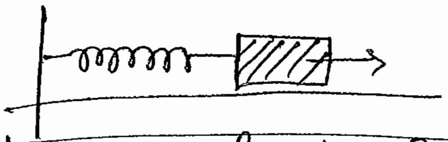
* Cyclic Coordinate / ignorable / removable coordinate :-

If Lagrangian does not depend on a coordinate then that coordinate is called cyclic coordinate.

Ex.- For a particle is moving in 3-D space (x, y, z).

$$\boxed{L = f(\dot{x}, \dot{y}, \dot{z}, x, y, t)}$$

z = is cyclic



$$\boxed{L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2}$$

No need to say that y and z are cyclic.

* Conservation theorem / Principle :-

If the system is conservative (monogenic).

Potential can be written for all forces.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

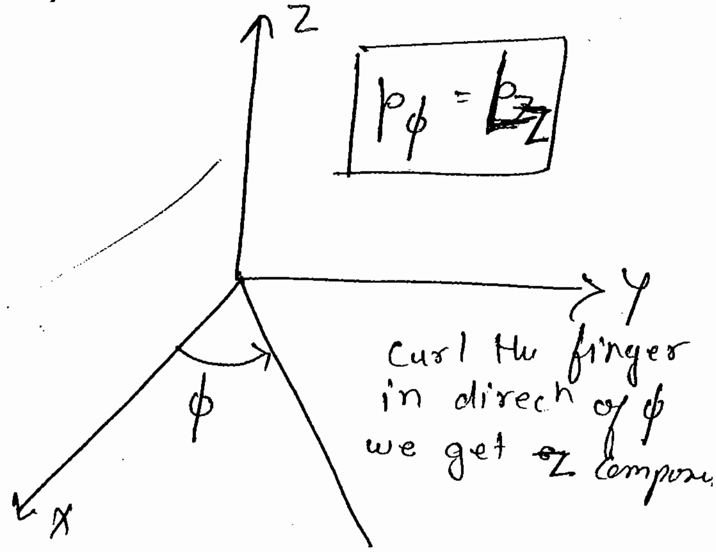
If L is no function of q_i (Cyclic)

$$\frac{\partial L}{\partial q_i} = 0$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$\rightarrow \boxed{\frac{\partial L}{\partial \dot{q}_i} = \text{Constant}} \Rightarrow \boxed{p_i = \text{Constant}}$$

Generalised momentum corresponding (conjugate) to cyclic generalised coordinate is conserved.
If x is cyclic then p_x is conserved or if ϕ (in spherical polar coordinate) is cyclic then p_ϕ = constant. $\{ p_\phi = L_z \}$.



* Problem Based On Lagrangian :-

$$L = T - V$$

$$\text{Equation of motion} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

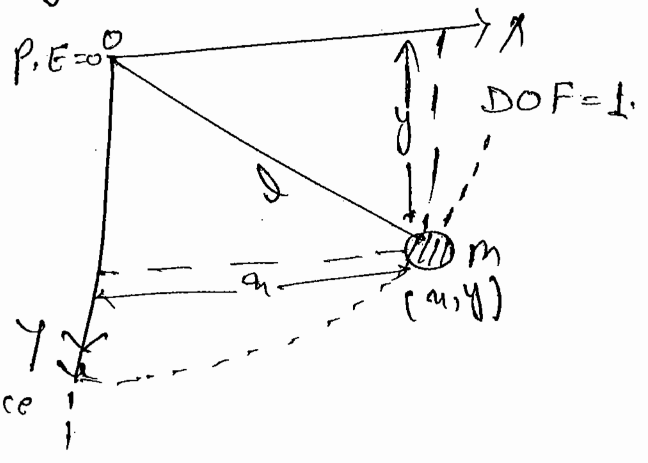
① Simple Pendulum :-

① Write Lagrangian in terms of x coordinate taken as shown in fig.

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$V = -mgy$$

Here -ve sign comes becoz of particle below the reference level.



$$L = T - V$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + mgy$$

$\because$ $x^2 + y^2 = l^2$

$\therefore$ $y^2 = l^2 - x^2$

$$\boxed{y = \sqrt{l^2 - x^2}}$$

$$\dot{y} = \frac{1}{2\sqrt{l^2 - x^2}} (-2x\dot{x}) \quad \{\text{Diff. w.r.to time}\}$$

$$\boxed{\dot{y} = \frac{-x\dot{x}}{\sqrt{l^2 - x^2}}}$$

So $L = \frac{1}{2} m \dot{x}^2 \left[1 + \frac{x^2}{l^2 - x^2}\right] + mg\sqrt{l^2 - x^2}$

$$\boxed{L = \left[\frac{1}{2} m \dot{x}^2 \left\{1 + \frac{m\dot{x}^2 l^2}{l^2 - x^2} + mg\sqrt{l^2 - x^2}\right]}$$

Here $L = f(x)$ $\qquad$ x is not cyclic

$\Rightarrow$ p_x is not conserved.

To know whether p_y is conserved or not we must express L in terms of y.
(becoz y is also changing when pendulum moves).

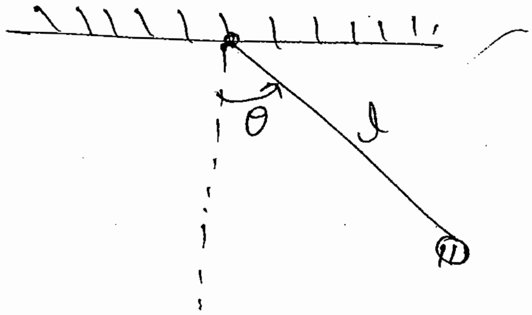
$\Rightarrow$ As pendulum oscillates in x-y plane only then z component of coordinate can not change It can not be considered $\therefore$ z is cyclic

$\therefore$ p_z is conserved.

* Simple Pendulum in terms of θ Coordinate:

We use here plane polar coordinate

PE = 0



DOF = 1

$$q = \theta$$

$$T = \frac{1}{2}(\dot{s}^2 + s^2\dot{\phi}^2)$$

$s = l =$ Constant $\therefore \dot{s} = 0$

$$\phi = \theta$$

$$\therefore T = \frac{1}{2}m(l^2 \dot{\theta}^2)$$

$$V = mgl\cos\theta$$

$$\therefore \boxed{L = \frac{1}{2}m(l^2\dot{\theta}^2) + mgl\cos\theta}$$

θ is not cyclic as angular momentum is not Conserved.

Equation of motion -

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}(ml^2\dot{\theta}) - mgl\sin\theta = 0$$

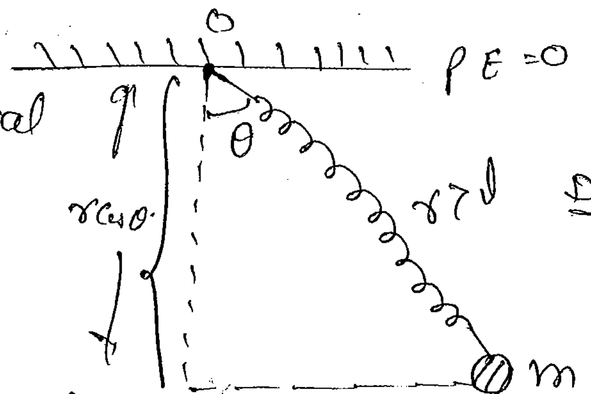
$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

Q. A simple pendulum consists of a small bob of mass 'm' suspended from light spring of ~~natural length 'l'~~ with spring constant 'k'. Write Lagrangian of the system and eq^n of motion using plane polar coordinate (r, θ). Natural length of spring is l.

Sol^n Case :- string ⟶ spring

It is not a vertical oscillation it is a planar oscillation



DOF = 2

spring doesnot put any constraint.

$$T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = mgr\cos\theta + \frac{1}{2} k (r-l)^2$$

$$L = T - V$$

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta - \frac{1}{2}k(r-l)^2$$

Equation of motion :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$q_1 = r, \quad q_2 = \theta$$

$$\frac{\partial q_i}{\partial q_j} = \delta_{ij}, \quad \frac{\partial q_1}{\partial q_2} = 0$$

r - quation $\rightarrow$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 \cancel{- mg} \cos\theta + K(r-l) = 0$$

$$\boxed{\ddot{r} - r\dot{\theta}^2 - g\cos\theta + \frac{k}{m}(r-l) = 0} \longrightarrow (*)$$

θ - quatron $\rightarrow$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}(mr^2\dot{\theta}) + mgr\sin\theta = 0$$

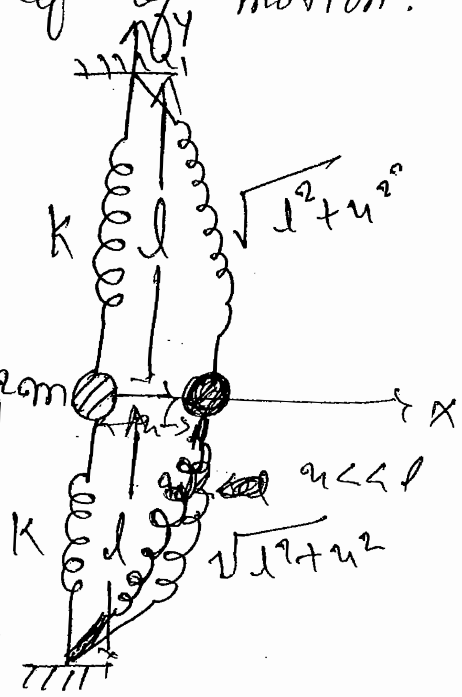
$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} + gr\sin\theta = 0$$

divide by r

$$\boxed{2\dot{r}\dot{\theta} + r\ddot{\theta} + g\sin\theta = 0} \longleftarrow (**)$$

NET-2013

Q. If the particle is slightly displaced along x-axis write its eqn of motion.



$\boxed{DOF = 1}$

Elongation $= \sqrt{l^2+x^2} - x$

$L = T - V$

$= \frac{1}{2}m\dot{x}^2 - \cancel{2}\frac{1}{\cancel{2}}K\left[\sqrt{l^2+x^2}-l\right]^2$

$$\boxed{L = \frac{1}{2}m\dot{x}^2 - K\left[\sqrt{l^2+x^2}-l\right]^2}$$

Equation of motion :- Here $\boxed{q = x}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt}[m\dot{x}] + 2k\left(\sqrt{l^2+x^2} - l\right)\cdot \frac{(\cancel{2}x)}{\cancel{2}\sqrt{l^2+x^2}} = 0$$

$$\Rightarrow m\ddot{x} + 2k\left(\sqrt{l^2+x^2} - l\right)\cdot \frac{x}{\sqrt{l^2+x^2}} = 0$$

$$\Rightarrow m\ddot{x} + 2k\left[1 - \frac{l}{\sqrt{l^2+x^2}}\right]x = 0$$

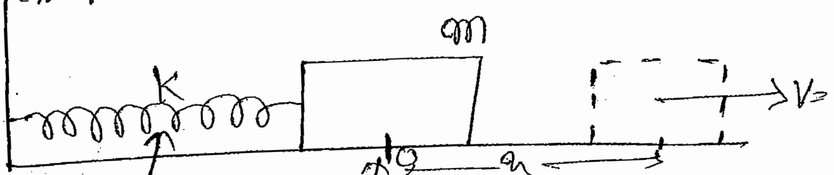
$$\Rightarrow \boxed{m\ddot{x} + 2Kx\left[1 - \frac{l}{\sqrt{l^2+x^2}}\right] = 0}$$

$$\Rightarrow m\ddot{x} + 2kx\left[1 - \left(1 + \frac{x^2}{l^2}\right)^{-1/2}\right] = 0 \quad \left\{\because (1+x)^n = (1+nx) \text{ neglect higher term}\right.$$

$$\Rightarrow m\ddot{x} + 2kx\left[1 - \left(1 - \frac{x^2}{2l^2}\right)\right]$$

$$\Rightarrow \boxed{m\ddot{x} + \frac{Kx^3}{l^2} = 0} \longleftarrow \text{Eq}^n \text{ of motion for small } x.$$

* Spring Mass System :-



Here elongation = x because the spring is oscillating along parallel to its length.

Relaxed (no elongation or Compression) mean position

So $L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} K x^2$

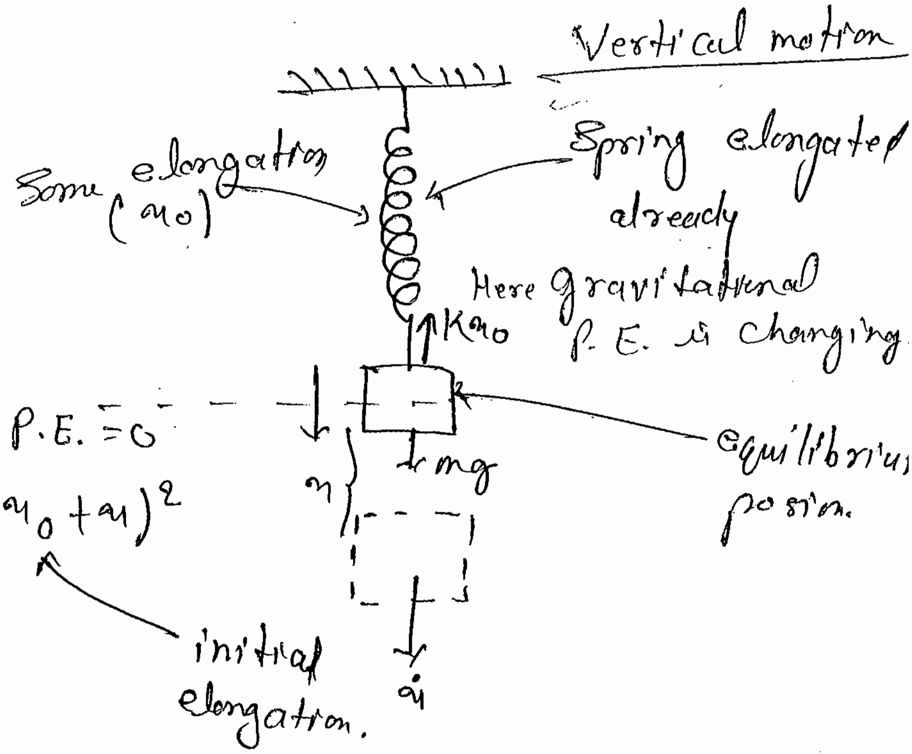
x = displacement from mean position.

Equation of motion :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\boxed{m\ddot{x} + kx = 0}$$

Q.



$L = T - V$

$T = \frac{1}{2} m \dot{x}^2$

$V = -mgx + \frac{1}{2}k(x_0 + x)^2$

$$L = \frac{1}{2} m\dot{x}^2 - \left[-mgx + \frac{1}{2}k(x_0+x)^2\right]$$

$$\boxed{L = \frac{1}{2} m\dot{x}^2 + mgx - \frac{1}{2}k(x+x_0)^2}$$

Equation of motion :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$m\ddot{x} - mg + k(x+x_0) = 0$$

$$\boxed{m\ddot{x} + kx - mg + kx_0 = 0} \quad (1)$$

At equilibrium position -

$$mg = Kx_0 \quad \text{—(II)}$$

from (I) and (II)

$$\boxed{m\ddot{x} + Kx = 0} \longrightarrow \text{Equation of motion at Equilibrium position.}$$

x is displacement from mean position.

Note :- "For Spring mass system if we can neglect initial elongation and gravitational potential energy then also we get correct equation of motion."

NET -2013

Q. A particle moving in a potential

$$V(x, y, z) = x^2 + y^2 + \frac{z^2}{2}$$

If L_x, L_y and L_z be Component of angular momentum then which is Constant or Conserved.

(a) L_x (b) L_y (c) L_z (d) None of these.

Soln Let us write Lagrangian in spherical polar coordinate.

$$L = T - V$$

$$\boxed{L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) - \left[r^2\sin^2\theta + \frac{r^2\cos^2\theta}{2}\right]}$$

$\because$ $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$\because$ ϕ is not coming in L so ϕ is cyclic

So p_ϕ is conserved. $\parallel$ $\boxed{L_z \text{ is Conserved}}$ Ans

* Lagrangian of a system is -

$$\boxed{L(x,\dot{x}) = \int_0^{\dot{x}} e^{-\alpha^2} d\alpha + \int_0^{x} e^{\alpha^2} d\alpha}$$

find eqn of motion.

Soln Eqn of motion:-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(e^{-\dot{x}^2}\right) - e^{x^2} = 0$$

$$e^{-\dot{x}^2}\cdot[-2\dot{x}\ddot{x}] - e^{x^2} = 0$$

$$\boxed{2\dot{x}\ddot{x}e^{-\dot{x}^2} + e^{x^2} = 0}$$

TIFR 2014

Q. If $f(x) = \int_0^\infty f(t)\,dt$ plot $f(x)$

$$\frac{df(x)}{dx} = f(x)$$

$$\int \frac{df(x)}{f(x)} = \int dx$$

$$\log f(x) = x + C$$

$$\boxed{f(x) = Ke^{x}}$$

* Lagrangian Dynamics:-

Spherical Polar Co-ordinate :-

$$T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2)$$

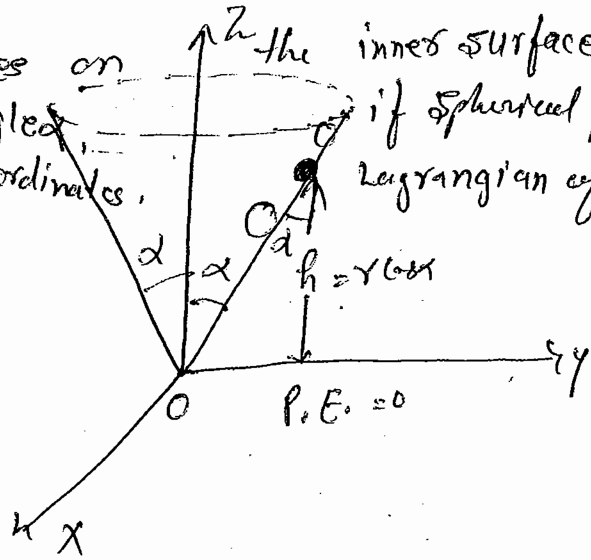
In Cylindical Co-ordinate -

$$T = \frac{1}{2} m (\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2)$$

Plane Polar :-

$$T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2)$$

A-10
Q.9 A particle of mass 'm' moves on the inner surface of inverted cone of half vertex angle α, if spherical polar co-ordinate, are taken as generalised co-ordinates, Lagrangian of the system particle is -



Spherical polar (r, θ, ϕ)

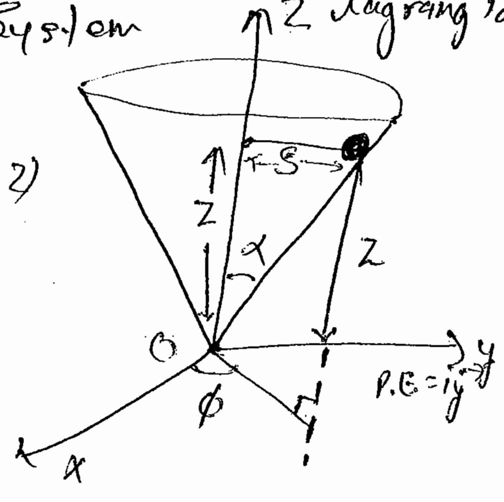
$\theta = \alpha \quad , \phi = 0$

$\dot{\theta} = 0$

$L = T - V$

$$\boxed{L = \frac{1}{2} m (\dot{r}^2 + r^2 \sin^2\alpha\, \dot{\phi}^2) - mgr\cos\alpha}$$

A-10
Q.10 In cylinderical co-ordinate system Lagrangian of bead in previous ~~q~~ question is -



Cylindrical Coordinate (s, ϕ, z)

$\tan\alpha = \frac{s}{z}$

$s = z\tan\alpha$

$$\dot{s} = \dot{z}\tan\alpha$$

$$L = T - V$$

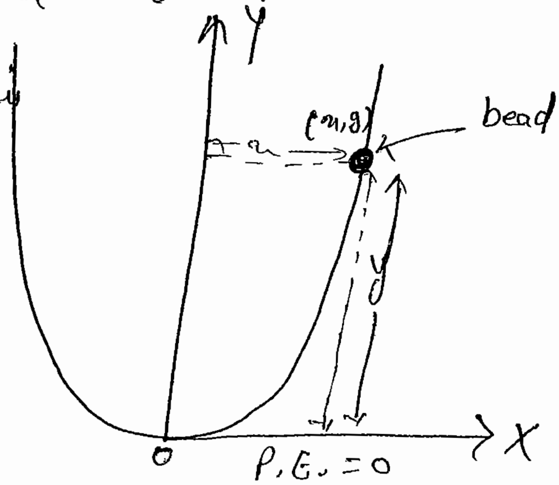
$$L = \frac{1}{2}m(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2) - mgz$$

$$\boxed{L = \frac{1}{2}m(\dot{z}^2\tan^2\alpha + z^2\tan^2\alpha\,\dot{\phi}^2 + \dot{z}^2) - mgz}$$

A-10

Q.1 A bead of mass 'm' slides along a wire kept in vertical pla as shown in fig. The eq^n of wire is $y = \alpha x^2$. Lagrangian of the bead is - ?

$y = \alpha x^2$



Lagrangian of bead

$\therefore \dot{y} = 2\alpha x\dot{x}$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$= \frac{1}{2}m(\dot{x}^2 + 4\alpha^2x^2\dot{x}^2) - mg\cdot\alpha x^2$$

$$\boxed{L = \frac{1}{2}m\dot{x}^2(1 + 4\alpha^2x^2) - mg\alpha x^2}$$

Write equation of motion :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left[m\dot{x}(1 + 4\alpha^2x^2)\right] - 4\alpha^2m\dot{x}^2x + 2mg\alpha x = 0$$

$$\ddot{x}(1 + 4\alpha^2x^2) + 8\alpha^2x\dot{x}^2 - 4\alpha^2\dot{x}^2x + 2mg\alpha x = 0$$

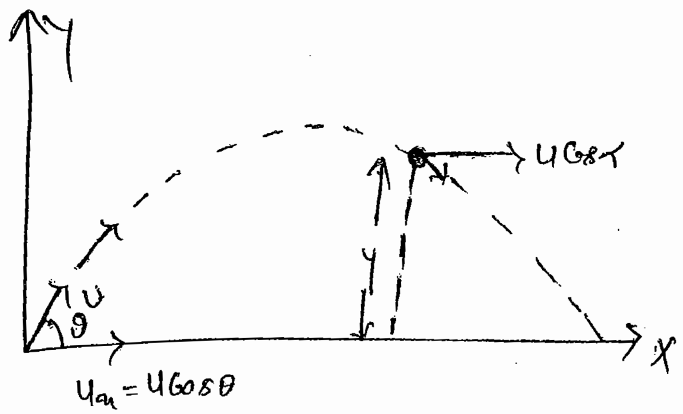
$$\boxed{\ddot{x}(1 + 4\alpha^2x^2) + 4\alpha^2x\dot{x}^2 + 2mg\alpha x = 0}$$

<u>Note</u> :- "Lagrangian is always written at any instant of time."
In Lagrangian initial velocity does not consider.

A-10
Q.2

$L = T - V$

$$\boxed{L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy}$$



Write eq^n of motion :-

x - equation :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$m\ddot{x} - 0 = 0$$

$$\ddot{x} = 0$$

$$\boxed{\dot{x} = \text{Constant}}$$

y - equation :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$m\ddot{y} + mg = 0$$

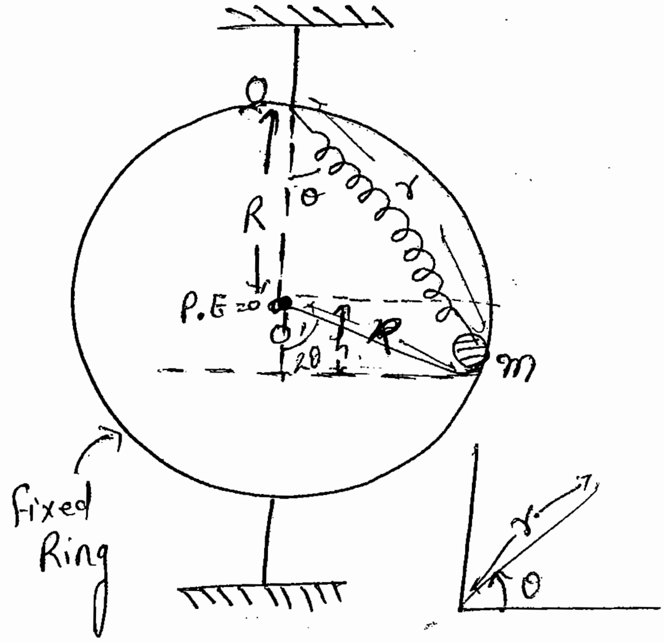
$$m(\ddot{y} + g) = 0$$

$$\ddot{y} + g = 0$$

$$\boxed{\ddot{y} = -g}$$

A-10

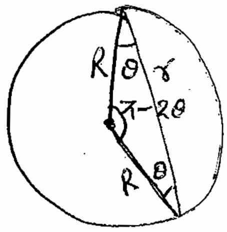
Q.4



$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = -mgh + \frac{1}{2} k(r - 2R)^2$$

use Sine rule

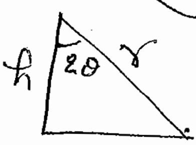


$$\frac{R}{\sin\theta} = \frac{r}{\sin(\pi - 2\theta)}$$

$$\cancel{\frac{R}{\sin\theta}} = \frac{r}{\sin 2\theta} = \frac{r}{2\cancel{\sin\theta}\cos\theta}$$

$$\therefore \quad r = 2R\cos\theta$$

$$\dot{r} = 2R\sin\theta.\dot{\theta}$$



$$h = R\cos 2\theta$$

$$L = T - V = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + mgh - \frac{1}{2} k(r - 2R)^2$$

$$= \frac{1}{2} m\left(4R^2\sin^2\theta.\dot{\theta}^2 + 4R^2\cos^2\theta.\dot{\theta}^2\right) + (mg.R\cos 2\theta) - \frac{1}{2} k(2R\cos\theta - 2R)^2$$

$$= \frac{1}{2} m(4R^2\dot{\theta}^2) + (mgR\cos 2\theta) - \frac{1}{2} k(4R^2\cos^2\theta + 4R^2\ 8R^2\cos\theta)$$

$$= \frac{1}{2} m(4R^2\dot{\theta}^2) + mgR\cos 2\theta - 2kR^2\cos^2\theta + 2R^2k + 4R^2k\cos\theta$$

$$= 2mR^2\dot{\theta}^2 + mgR\cos 2\theta - 2kR^2\cos^2\theta + 2R^2k + 4R^2k\cos\theta$$

$$\boxed{L = 2mR^2\dot{\theta}^2 - 2kR^2(1 - \cos\theta)^2 + mgR\cos 2\theta}$$

Ans

Q.5 In previous question, equation of motion of bead for small value of θ is ?

~~(a)~~ $\ddot{\theta} + \frac{g}{R}\theta = 0$ (b) $\ddot{\theta} + \frac{2g}{R}\theta$

(c) $\ddot{\theta} + \frac{g}{2R}\theta = 0$ (d) $\ddot{\theta} + \frac{4g}{R}\theta = 0$

Sol^n

for small values of θ

$\sin\theta = \theta$

$\cos\theta = 1$

$$L = 2mR^2\dot{\theta}^2 - 2kR^2(1-\cos\theta)^2 + mgR\cos 2\theta$$

$$L = 2mR^2\dot{\theta}^2 + mgR\cos 2\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) + \frac{\partial L}{\partial \theta} = 0$$

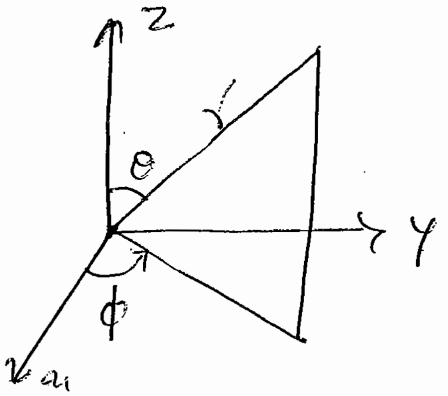
$$\Rightarrow 4mR^2\ddot{\theta} + mgR \cdot 2\sin 2\theta = 0$$

$$4mR^2\ddot{\theta} + 4mgR\theta = 0$$

$$4mR^2\left(\ddot{\theta} + \frac{g}{R}\theta\right) = 0$$

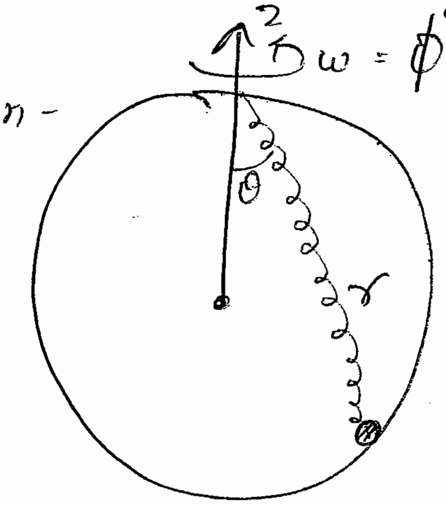
$$\boxed{\ddot{\theta} + \frac{g}{R}\theta = 0}$$ Ans

* If the ring is rotated with angular speed ω as shown :-



Write Lagrangian -

$\sin\theta \approx \theta$
$\cos\theta \approx 1$



$p_\phi = L_z$

ϕ represents rotation about z axis (on spherical & cylindrical coordinate)

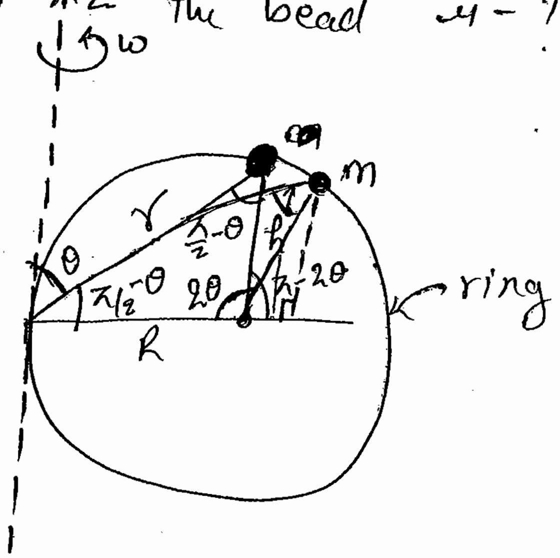
$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\, \dot{\phi}^2\right)$$

$$\boxed{T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\, \omega^2\right)}$$

A-10

Q.11 A bead of mass 'm' is sliding on a vertical circular loop of radius R. The loop is rotated with constant angular velocity ω about a tangential axis shown in figure. Lagrangian of the bead is - ?

Sol^n



$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2)$$

$$V = mgh = mgR \sin(\pi - 2\theta)$$

$$V = mgR \sin 2\theta$$

Use sine Rule

$$\frac{R}{\sin(\frac{\pi}{2} - \theta)} = \frac{r}{\sin 2\theta}$$

$$\boxed{r = 2R \sin\theta}$$

$$\dot{\phi} = \omega,$$

$$\dot{r} = 2R \cos\theta \, \dot{\theta}$$

$$T = \frac{1}{2} m\left(4R^2 \cos^2\theta \, \dot{\theta}^2 + 4R^2 \sin^2\theta \, \dot{\theta}^2 + 4R^2 \sin^4\theta \, \omega^2\right)$$

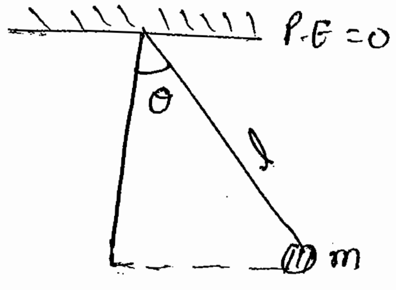
$$T = \frac{1}{2} m(4R^2\dot{\theta}^2 + 4R^2 \sin^4\theta \, \omega^2)$$

$$L = T - V$$

* Simple Pendulum :-

Oscillations is in plane. $r = l$

$\dot{r} = 0$



$$\boxed{L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta}$$

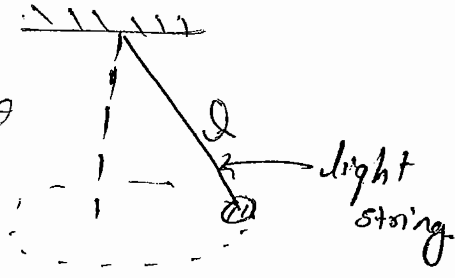
* Spherical Pendulum :-

Plane of oscillation is not fixed (then is oscillation about z-axis)

Use spherical polar.

$r = l$, θ $\dot{r} = 0$

$$L = \frac{1}{2} m \left(\dot{\theta}^2 l^2 + l^2 \sin^2\theta\, \dot{\phi}^2\right) + mgl\cos\theta$$

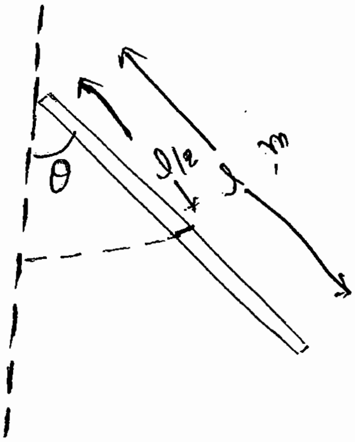


$$\boxed{L = \frac{1}{2} m l^2\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right) + mgl\cos\theta}$$

* A thin rod suspended from one end:

$$L = \frac{1}{2}\left(\frac{ml^2}{3}\right)\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right) + mg\frac{l}{2}\cos\theta$$

$$\boxed{L = \frac{-ml^2}{6}\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right) + \frac{mgl\cos\theta}{2}}$$



* A rod is suspended from middle (free to rotate in any orientation):

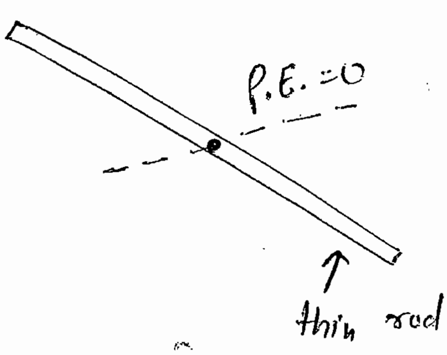
Centre of is not moving.

$K = \frac{1}{2} I\omega^2$

$K = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega}$

$$L = \frac{1}{2}\left(\frac{ml^2}{12}\right)\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right)$$

$$\boxed{L = \frac{ml^2}{24}\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right)}$$



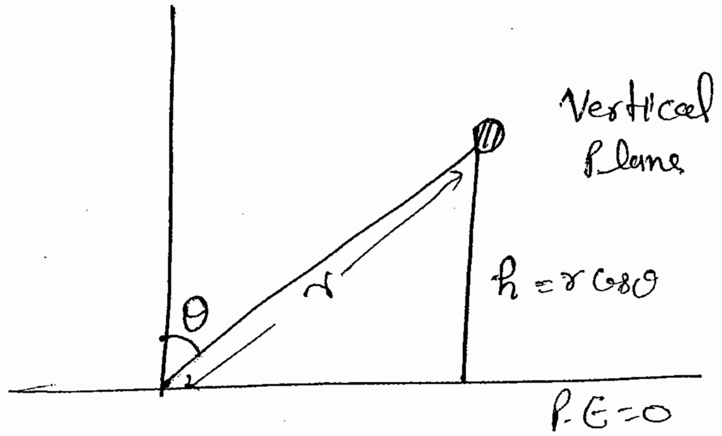
A-10

Q.12

Sol^n

∵ Here θ and h both variable

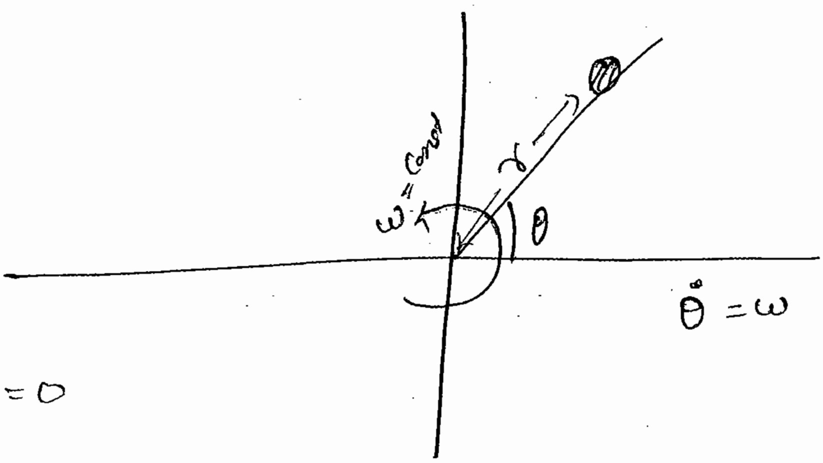
∵ It is plane polar co-ordinate



So $\boxed{L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta}$

A-10
Q.13

Sol^n



$$L = \frac{1}{2} m(\dot{r}^2 + r^2\omega^2)$$

r - eq^n

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$m\ddot{r} - mr\omega^2 = 0$$

$$\ddot{r} - r\omega^2 = 0$$

$$\frac{d^2r}{dt^2} - \omega^2 r = 0$$

$$(D^2 - \omega^2)r = 0$$

roots $= \pm\omega$

$\boxed{r = Ae^{\omega t} + Be^{-\omega t}}$ Ans

A-10 F.

Q.25 Lagrangian of a system is $L = a\dot{x}^2 - bx^2$ then

(a) $x = c_1 t + c_2 t^2 + c_3$ (b) $x = c_1 e^{-c_2 t} + c_3$

(c) $x = c_1 \sin(c_2 t + c_3)$ (d) $x = c_1 e^{-c_2 t} \sin(c_2 t + c_3)$

Soln

Equation of motion -

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$2a\ddot{x} + 2bx = 0$$

$$\ddot{x} + \frac{b}{a}x = 0$$

$$\boxed{\text{roots} = \pm i\sqrt{\frac{b}{a}}}$$ Purely imaginary.

$$\boxed{x = c_1 \sin(c_2 t + c_3)}$$

B-A-2

Q.47 NET 2012 Dec.

Soln

$$L = \frac{1}{2}m\dot{x}^2 - bx$$

Equation of motion:-

$$m\ddot{x} + b = 0$$

$$\ddot{x} = -\frac{b}{m}$$

$$\frac{d^2x}{dt^2} = -\frac{b}{m}$$

Integrate -

$$\frac{dx}{dt} = -\frac{b}{m}t + c_1$$

$$\boxed{x = -\frac{b}{m}t^2 + c_1 t + c_2}$$

Ans

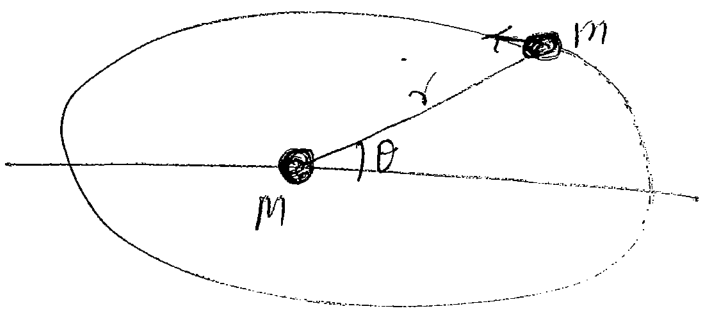
Imp. #10

Q.24 A planet of mass 'm' revolves around the Sun of mass M in an elliptical orbit. If motion of planet is confined in one plane, Lagrangian of the system is -?

(a) $\frac{1}{2}(\dot{r}^2+r^2\dot{\theta}^2)+\frac{GMm}{r}$ (b)

(c) (d)

Sol^n



$$T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)$$

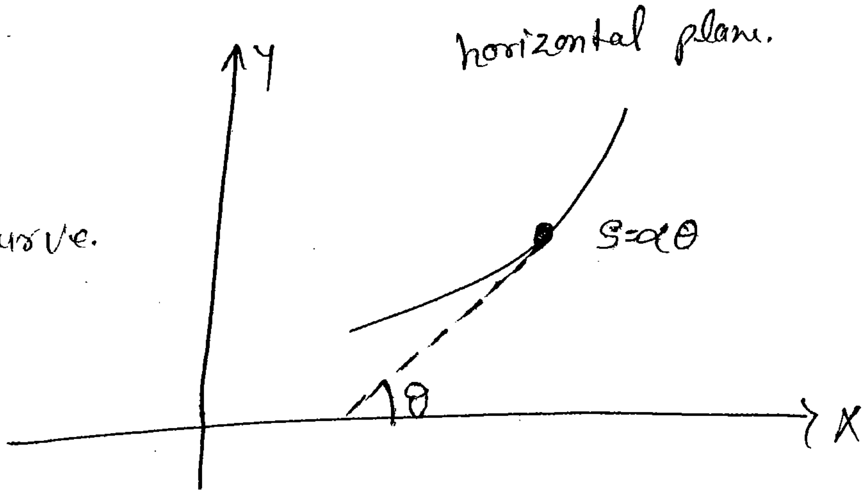
$$V=-\frac{GMm}{r}$$

$$L=T-V$$

$$\boxed{L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+\frac{GMm}{r}}$$

Q.22

Sol^n



Equation of curve

s = dist along curve.

$$L=\frac{1}{2}mv^2 \quad -①$$

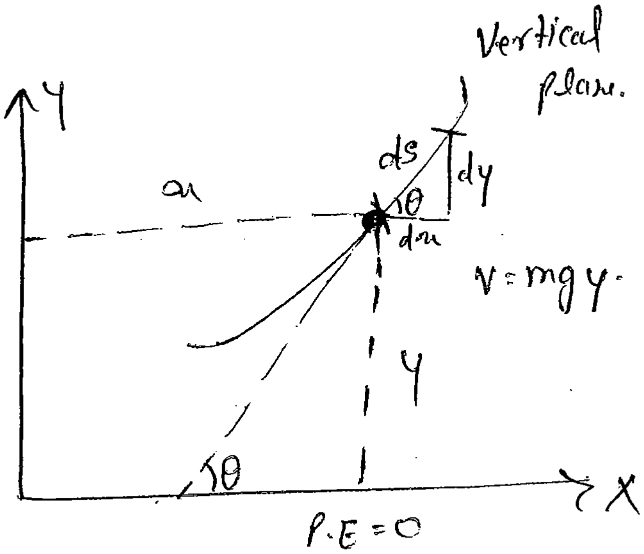
$$=\frac{1}{2}m\left(\frac{ds}{dt}\right)^2$$

$\because \; s = \alpha\theta$

$$\boxed{L = \frac{1}{2} m \alpha^2 \dot{\theta}^2} \qquad \frac{ds}{dt} = \alpha\dot{\theta}$$

Q. In previous question plane is vertical write lagrangian in θ co-ordinate.

Sol^n



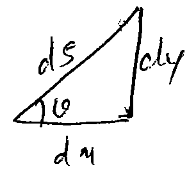
$L = T - V \qquad V = mgy$

$L = \frac{1}{2} m\alpha^2 \dot{\theta}^2 + mg\alpha\cos\theta$

$\sin\theta = \frac{dy}{ds}$

$s = \alpha\theta$

$ds = \alpha d\theta$



$\sin\theta = \frac{dy}{\alpha d\theta}$

$\int dy = \int \alpha \sin\theta d\theta$

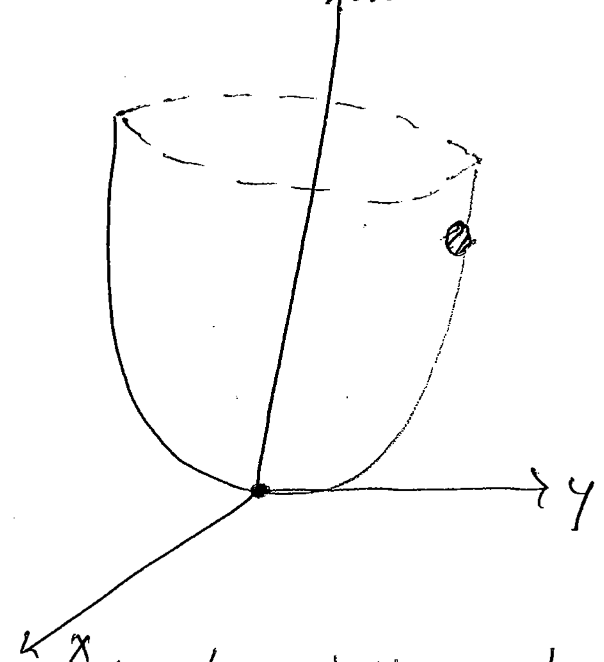
$\boxed{y = -\alpha\cos\theta}$

NET-2011

Q.38 A particle of mass 'm' moves inside a bowl. If the surface of the bowl is given by the equation $z = \frac{1}{2} a(x^2+y^2)$, where 'a' is a constant, the Lagrangian of the particle is.

Sol^n

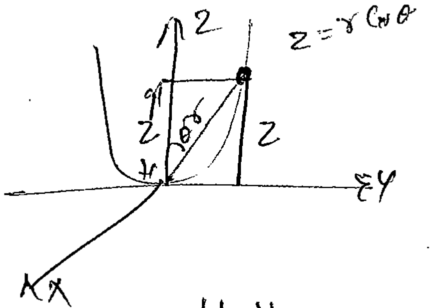
$\boxed{z = \frac{1}{2} a (x^2 + y^2)}$



a = Constant.

He co-ordinate is not given so we check which co-ordinate system we use -

When we ~~see~~ option we confused whether it is spherical ~~pto~~ polar of cylindrical ~~po~~ co-ordinate.
So we check potential energy in both region to identify the co-ordinate system.



$V = mgz$

$\boxed{V = mgr\cos\theta}$ not match

Whether we check $z = \frac{1}{2} a (x^2 + y^2)$ given

So $z = \frac{1}{2} a r^2 \sin^2\theta$ where $x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$

which is also mismatch.

So we check in cylindrical co-ordinate:-

$V = mgz$

$x = r\cos\phi$
$y = r\sin\phi$
$z = z$

So $\boxed{V = mg\frac{1}{2}ar^2}$

match in 3 options.

Here options are independent of z so we have to remove z.

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right)$$

$$z = \frac{1}{2} a r^2$$

$$\dot{z} = a r \dot{r}$$

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\phi}^2 + a^2 r^2 \dot{r}^2\right)$$

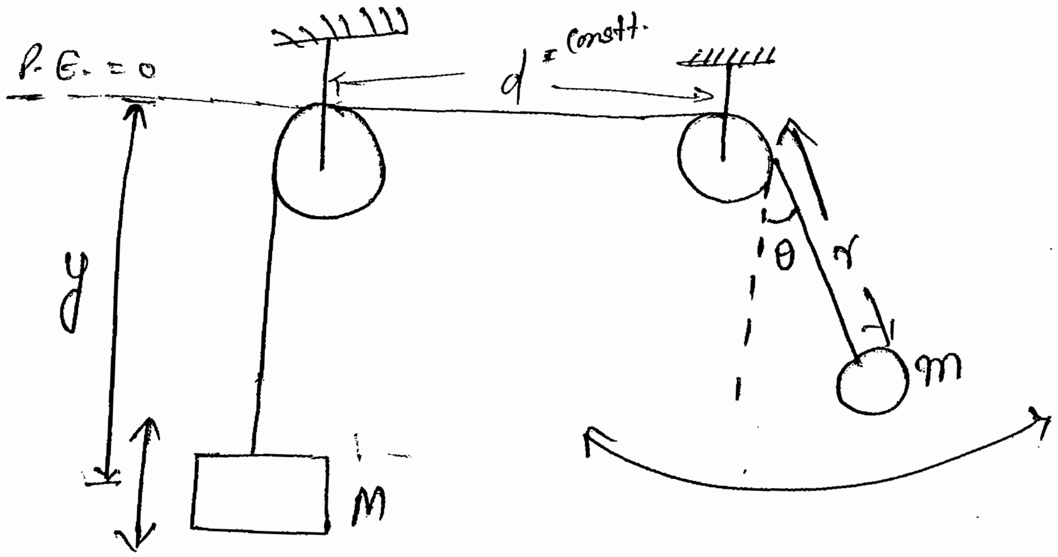
$$\boxed{T = \frac{1}{2} m \left(\dot{r}^2(1 + a^2 r^2) + r^2\dot{\phi}^2\right)}$$

$$L = T - V$$

$$L = \frac{1}{2} m \left(\dot{r}^2(1+a^2r^2) + r^2\dot{\phi}^2\right) - \frac{1}{2} m g a r^2$$

$$\boxed{L = \frac{1}{2} m \left(\dot{r}^2(1+a^2r^2) + r^2\dot{\phi}^2 - g a r^2\right)}$$ Ans

Q.



Write the eq^n of motion of the two bloqs.

Sol^n DOF = 2.

Length of string is constant. if it is inextensible

$y + d + r =$ Constant $=$ length of string $= l$.

$$\dot{y} + 0 + \dot{r} = 0$$

$$\boxed{\dot{y} = -\dot{r}}$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2} m \dot{y}^2$$

$$\boxed{T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2} m \dot{r}^2}$$

$$V = -Mgy - mgr \cos\theta$$

$$\boxed{V = -Mg(l - d - r) - mgr\cos\theta.}$$

$$\boxed{L = \frac{1}{2}\left[(m + M)\dot{r}^2 + mr^2\dot{\theta}^2\right] + Mg(l - d - r) + mgr\cos\theta.}$$

Eqn of motion -

r - eqn :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

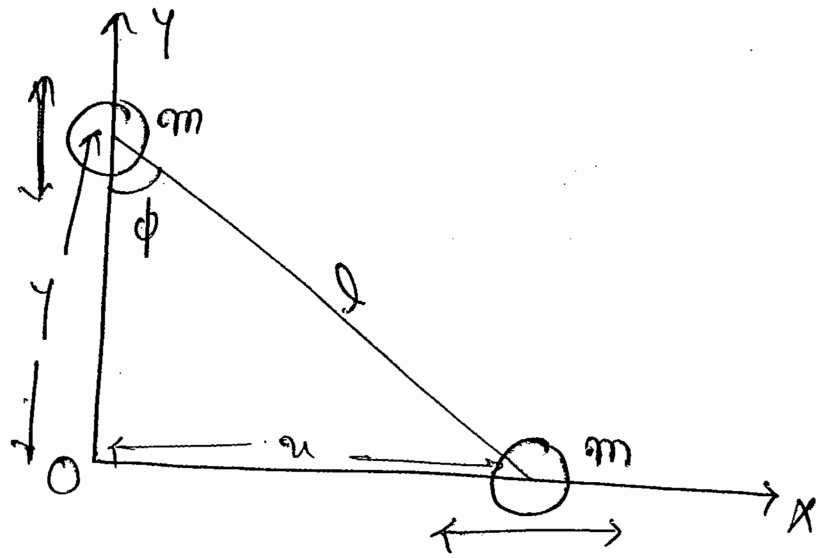
$$\boxed{(m+M)\ddot{r} + Mg - Mg\cos\theta - mr\dot{\theta}^2 = 0}$$ eqn of motion.

$$\cancel{\ddot{r} = g(m\cos\theta - M)}$$

Q. Two particle connected by a light rod. Particles are constrained to move along two ⊥^{or} lines. If ϕ is generalied co-ordinate What is K.E. of the system.

Sol^n

$\boxed{DOF = 1}$



$$K.E. = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2$$

$$K.E. = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

from Δ

$$x = l \sin\phi \Rightarrow \dot{x} = l \cos\phi \, \dot{\phi}$$

$$y = l \cos\phi \Rightarrow \dot{y} = -l \sin\phi \, \dot{\phi}$$

So $\boxed{K.E. = \frac{1}{2} m l^2 \dot{\phi}^2}$

* Small Oscillation :-

Normal Modes :

It is a special type of oscillation in which all particles of system oscillates with same frequency.

Finding frequency of Normal modes :-

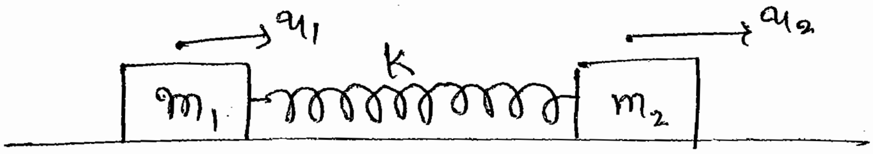
It is obtained by solving following matrix equation or determinant.

$$\left| \hat{V} - \omega^2 \hat{T} \right| = 0$$

Where $\hat{V}$ is a matrix obtained from potential ~~kinetic~~ energy.
$\hat{T}$ " " " " kinetic "

To obtain $\hat{V}$ we take cofficients of q_i, q_j in P.E.
" $\hat{T}$ " " " K.E.

Q. Two masses connected by a light spring. masses are free to oscillate.



find frequency (angular) of Normal modes?

Sol^n DOF = 2.

Let x_1 and x_2 be displacement of m_1 and m_2 from their mean position

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2}\left(m_1 \dot{x}_1 \dot{x}_1 + m_2 \dot{x}_2 \dot{x}_2\right)$$

$\therefore \quad \hat{T} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ { Here $\frac{1}{2}$ is not consider becoz it is canceled out finally.

$$V = \frac{1}{2} K (x_2 - x_1)^2 = \frac{1}{2} K \left(x_1^2 + x_2^2 \cancel{+} 2 x_1 x_2\right)$$

$$V = \frac{1}{2}\left(K x_1^2 + K x_2^2 - K x_1 x_2 - K x_2 x_1\right)$$

$$\therefore \quad \hat{V} = \begin{pmatrix} K & -K \\ -K & K \end{pmatrix}$$

Secular Equation :-

$$\left| \hat{V} - \omega^2 \hat{T} \right|$$

$$\begin{vmatrix} K - m_1 \omega^2 & -K \\ -K & K - m_2 \omega^2 \end{vmatrix} = 0$$

$$(K - m_1 \omega^2)(K - m_2 \omega^2) - K^2 = 0$$

$$\cancel{K^2} - K(m_1+m_2)\omega^2 + m_1 m_2 \omega^4 - \cancel{K^2} = 0$$

$$\omega^2 \left[-K(m_1+m_2) + m_1 m_2 \omega^2 \right] = 0$$

$$\omega^2 = 0 \longrightarrow \omega = 0$$

$$\omega^2 = \frac{K(m_1+m_2)}{m_1 m_2}$$

$$\omega = \sqrt{\frac{K}{\left(\frac{m_1 m_2}{(m_1+m_2)}\right)}}$$

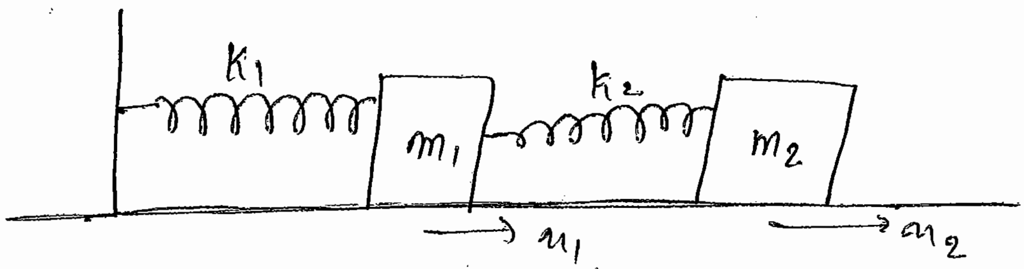
Here -ve term is not considered becoz freq. is never negative

$$\boxed{\omega = \sqrt{\frac{K}{\mu}}}$$

Where $\boxed{\mu = \frac{m_1 m_2}{m_1 + m_2}}$

<u>Note</u>: If system is not rigidly fixed then one of freq. comes out to be zero.

<u>Q.</u>



If x_1 and x_2 are displacement from mean positions.

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$V = \frac{1}{2} K_1 x_1^2 + \frac{1}{2} K_2 (x_2 - x_1)^2$$

$$\hat{T} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

$$\hat{V} = \begin{pmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{pmatrix}$$

<u>Note</u> :- If x and θ are involved in a question then to calculate ω we will consider $(l\theta)$ and x as two coordinates [$\because$ so that the two co-ordinates so that the two co-ordinates have same dimension.

* Approximation :-

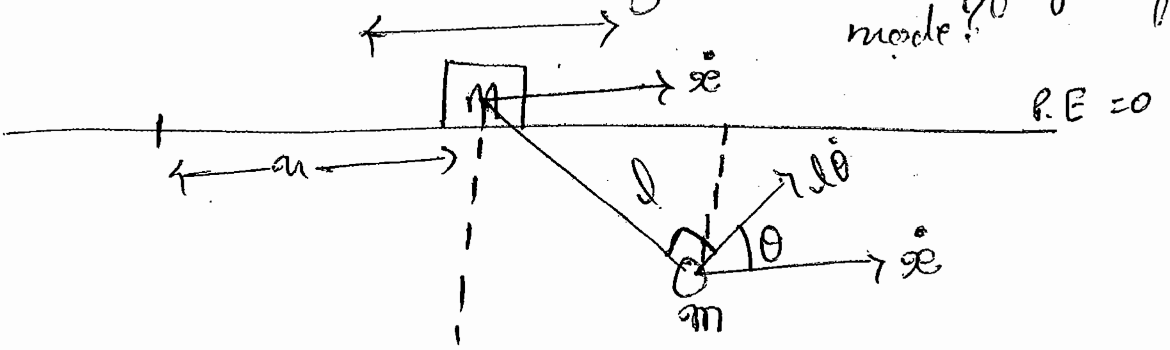
$$mgl\cos\theta \simeq mgl\left(1 - \frac{\theta^2}{2}\right)$$

$$\dot{x}\dot{\theta}\cos\theta \simeq \dot{x}\dot{\theta}\left(1 - \frac{\theta^2}{2}\right)$$

$$\simeq \dot{x}\dot{\theta} - \dot{x}\dot{\theta}\frac{\dot{\theta}^2}{2}$$ → very small ≈ 0 for small oscillation

$$\simeq \dot{x}\dot{\theta}$$

Q. Block can move forward and backward and pendulum is oscillating. What is freq. of normal mode?



$$T = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m\left[\dot{x}^2 + (l\dot{\theta})^2 + 2\dot{x}(l\dot{\theta})\cos\theta\right]$$

for small oscillations

$$T \simeq \frac{1}{2}\left[M\dot{x}^2 + m\left[\dot{x}^2 + (l\dot{\theta})^2 + 2\dot{x}(l\dot{\theta})\right]\right]$$

$x \rightarrow$ first coordinate

$l\theta \rightarrow$ Second Co-ordinate.

$$\hat{T} = \begin{pmatrix} M+m & m \\ m & m \end{pmatrix}$$

$$V = -mgl\cos\theta$$

$$= -mgl\left(1 - \frac{\theta^2}{2}\right) = -mgl + \frac{1}{2} mgl\,\theta^2$$

$$= -mgl + \frac{1}{2}\,\underline{\frac{mg}{l}}\,\underline{(l\theta)^2}$$

So $\hat{V} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{mg}{l} \end{pmatrix}$

So Secular equation -

$$\left|\hat{V} - \omega^2\hat{T}\right| = 0$$

$$\left|\omega^2\hat{T} - \hat{V}\right| = 0$$

$$\begin{vmatrix} \omega^2(M+m) & m\omega^2 \\ m\omega^2 & m\omega^2 + \frac{mg}{l} \end{vmatrix} = 0$$

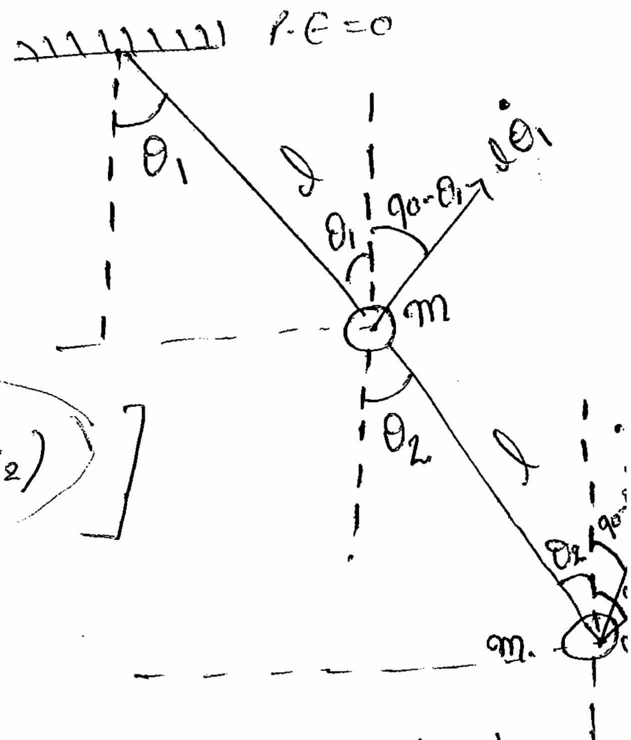
$$\omega^2(M+m)\cancel{m}\left(\omega^2 - \frac{g}{l}\right) - m^2\omega^4 = 0$$

$$\omega^2\left[(M+m)\left(\omega^2 - \frac{g}{l}\right) - m\omega^2\right] = 0$$

$$\Rightarrow \boxed{\begin{aligned} \omega &= 0 \\ \omega &= \sqrt{\frac{g}{l}\left(1+\frac{m}{M}\right)} \end{aligned}}$$

If $M \to \infty$ then we get result of simple pendulum.

Q. Write $\hat{T}$ and $\hat{V}$.



$$T = \frac{1}{2} m(l\dot{\theta}_1)^2 + \frac{1}{2} m\Big[(l\dot{\theta}_1)^2 + (l\dot{\theta}_2)^2 + 2(l\dot{\theta}_1)(l\dot{\theta}_2)\underbrace{\cos(\theta_1-\theta_2)}_{\approx 1}\Big]$$

$$\hat{T} = \begin{pmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{pmatrix}$$

θ_1 = first Co-ordinate

θ_2 = Second Co-ordinate.

$$V = -mgl\cos\theta_1 - mgl\left(\cos\theta_1 + \cos\theta_2\right)$$

$$= -mgl\left[2\cos\theta_1 + \cos\theta_2\right]$$

$$= -mgl\left[2\left(1 - \frac{\theta_1^2}{2}\right) + \left(1 - \frac{\theta_2^2}{2}\right)\right]$$

$$V = -3mgl + \frac{1}{2}\left[2mgl\,\theta_1^2 + mgl\,\theta_2^2\right]$$

$$\hat{V} = \begin{pmatrix} 2ml & 0 \\ 0 & mgl \end{pmatrix}$$

Hamiltonian Dynamics

A transition from Lagrangian function to Hamiltonian function.

$$L \xrightarrow[\text{Transformation}]{\text{Legendre}} H$$

$$\boxed{H(q_i, p_i) = p_i \dot{q}_i - L(q_i, \dot{q}_i)}$$

Hamiltonian.

In hamiltonian dynamics q_i & p_i are taken as independent variable.

$$\frac{\partial p_i}{\partial q_i} = 0 \quad , \quad \frac{\partial q_i}{\partial p_i} = 0$$

$\frac{\partial \dot{q}_i}{\partial q_i}$ may not be zero

$\frac{\partial \dot{q}_i}{\partial p_i}$ may not be zero.

$$\frac{\partial p_i}{\partial p_j} = \delta_{ij} \quad , \quad \frac{\partial q_i}{\partial q_j} = \delta_{ij}$$

* Hamilton's Equation of motion:-

It is observed from -

1. Legendre Transform
2. Variational principle.

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}}$$

$$\boxed{\dot{p}_i = \frac{-\partial H}{\partial q_i}}$$

* Dynamical Variable :- A function of any quantity which is function of q_i, p_i and t. If A is dynamical variable, Then -

$$A = A(q_i, p_i, t)$$

* Possion's Equations of Motion :- It gives the rate of change of a dynamical variable.

$$\frac{dA}{dt} = \frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i + \frac{\partial A}{\partial t}$$

$$= \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial A}{\partial p_i}\left(\frac{-\partial H}{\partial q_i}\right) + \frac{\partial A}{\partial t}$$

$$= \underbrace{\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i}}_{\text{Possion Bracket of } A \text{ with } H.} + \frac{\partial A}{\partial t}$$

$$\boxed{\frac{dA}{dt} = [A, H] + \frac{\partial A}{\partial t}}$$

This is possion's eqn of motion.

If A does not explicitly depends on time t.

$$\frac{\partial A}{\partial t} = 0$$

$$\frac{dA}{dt} = [A, H]$$

$$\boxed{\dot{A} = [A, H]}$$

If A is constant

$$\dot{A} = 0$$

$$\boxed{[A, H] = 0}$$

If a quantity is constant then its possion's Bracket with H is zero.

* Hamilton's Equation in terms of possion's bracket:-

$$\boxed{\begin{aligned} \dot{q}_i &= [q_i, H] \\ \dot{p}_i &= [p_i, H] \end{aligned}}$$ → Gate 2014

or

$$\boxed{\begin{aligned} [q_i, H] &= \frac{\partial H}{\partial p_i} \\ [p_i, H] &= -\frac{\partial H}{\partial q_i} \end{aligned}}$$

* Conversion of Lagrangian (L) into Hamiltonian (H):

Steps :-

(I) Use $p_i = \frac{\partial L(q_i, \dot{q}_i)}{\partial \dot{q}_i}$ and find $\dot{q}_i$

(II) Use $H = p_i \dot{q}_i - L$ and put the value of $\dot{q}_i$

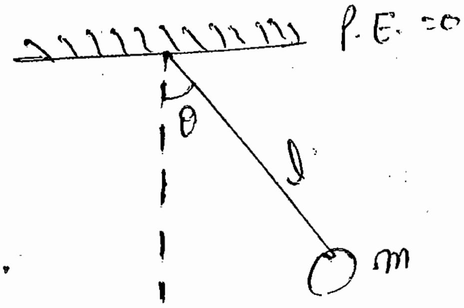
* Conversion of Hamiltonian (H) into Lagrangian (L) :-

Steps :-

(1) Use $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and find p_i

(2) Use $L = p_i \dot{q}_i - H$ and put value of p_i

Simple Pendulum



$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

Suppose it is given find H.

Soln

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\dot{\theta} = \frac{p_\theta}{ml^2}$$

$$H = p_\theta \dot{\theta} - L$$

So $H = p_\theta \dot{\theta} - \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta$

put $\dot{\theta}$

$$H = \frac{p_\theta^2}{ml^2} - \frac{1}{2}\frac{p_\theta^2}{ml^2} - mgl\cos\theta$$

$$\boxed{H = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta}$$

Net 2014 June

Q. $L = \frac{1}{2} m\dot{q}^2 - \frac{a}{2} q\dot{q}^2$ find H?

Sol^n

$$p = \frac{\partial L}{\partial \dot{q}}$$

$$p = m\dot{q} - a q\dot{q} = \dot{q}(m - aq)$$

$$\boxed{\dot{q} = \frac{p}{m - aq}}$$

$$H = p\dot{q} - L = p\dot{q} - \frac{1}{2} m\dot{q}^2 + \frac{a}{2} q\dot{q}^2$$

put $\dot{q}$

$$H = \frac{p^2}{m - aq} - \frac{1}{2}\dot{q}^2(m - aq)$$

$$= \frac{p^2}{m - aq} - \frac{1}{2}\frac{p^2}{(m - aq)^{\cancel{2}}} \cancel{(m - aq)}$$

$$\boxed{H = \frac{p^2}{2(m - aq)}}$$

Q $L = \frac{1}{2}mv^2 + \vec{a}\cdot\vec{v}$. Where $\vec{v}$ = velocity
$\vec{a}$ = Constant vector,
find $H = ?$

Sol^n Some mathematical methods -

$$\vec{a} \longleftrightarrow a_i$$

$$\vec{a}\cdot\vec{b} \longleftrightarrow a_i b_i$$ {when repetation of index then we assume Sumation}

$$\vec{a}^2 \longleftrightarrow a_i^2$$

Summation over repeated index (i) is assumed.

$$(\vec{a}\times\vec{b})_i = \underbrace{\varepsilon_{ijk}}_{\text{Levicivita Tensor}} a_j b_k$$

$$\varepsilon_{ijk} = \hat{i}\cdot(\hat{j}\times\hat{k})$$

if two index are equal then $\varepsilon = 0$

$$\varepsilon_{iik} = \hat{i}\cdot(\hat{i}\times\hat{k}) = 0$$

$$\varepsilon_{ijk} = -\varepsilon_{ikj}$$

$$\varepsilon_{ijj} = 0$$

$$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$$

Now come in question -

$$L = \frac{1}{2} m v_i^2 + a_i v_i$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial v_i}$$

generalised velocity

$$p_i = m v_i + a_i$$

$$v_i = \frac{p_i - a_i}{m}$$

$$H = p_i \dot{q}_i - L$$

$$= p_i v_i - \frac{1}{2} m v_i^2 - a_i v_i$$

$$H = p_i \left(\frac{p_i - a_i}{m}\right) - \frac{1}{2} \frac{(p_i - a_i)^2}{m} - \frac{a_i (p_i - a_i)}{m}$$

$$= \frac{(p_i - a_i)}{m}(p_i - a_i) - \frac{(p_i - a_i)^2}{2m}$$

$$= \frac{(p_i - a_i)^2}{2m}$$

$$\boxed{H = \frac{(\vec{p} - \vec{a})^2}{2m}}$$

* Some Standared Lagrangian and Corresponding Hamiltonian :-

① Free Particle ($V=0$)

(i) Non - Relativistic :-

$$L = \frac{1}{2} m v^2$$

$$H = \frac{p^2}{2m}$$

(ii) Relativistic Case :-

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}}$$ ← It gives correct expression for momentum.

$$H = \sqrt{p^2 c^2 + m_0^2 c^4}$$

② Particle moving in electromagnetic field (ϕ, $\vec{A}$)

ϕ ↑ Electric Potential; $\vec{A}$ ↑ mag. Vector poten.

(i) Non - Relativistic :-

$$L = \frac{1}{2} m v^2 - q\phi + q\vec{A}\cdot\vec{v}$$

$$H = \frac{(\vec{P} - q\vec{A})^2}{2m} + q\phi$$

$$P.E(V) = \underbrace{q\phi - q\vec{A}\cdot\vec{v}}$$

{q = charge.}

↓ It gives correct Lorentz force.

(ii) Relativistic Case :-

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} - q\phi + q\vec{A}\cdot\vec{v}$$

$$H = \sqrt{(\vec{P} - q\vec{A})^2 c^2 + m_0^2 c^4} + q\phi$$

$$L = T - V$$

$$L = mc^2 - m_0c^2$$

✗ $L = \dfrac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2 \longrightarrow$ We can not write Lagrangi[an] because it can not give correct expression for momentum.

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial v} = \frac{m_0 v^2}{(1-v^2/c^2)^{3/2}}$$ but this is not correct.

Correct expression for momentum.

$$p = mv. \quad \frac{m_0 v}{\sqrt{1-v^2/c^2}}$$

but $L = -m_0c^2\sqrt{1-v^2/c^2}$

$$\frac{\partial L}{\partial v} = p = \frac{m_0 v}{\sqrt{1-v^2/c^2}}$$ This is the correct expression of Lagrangian.

* Conversion of Lagrangian into Hamiltonian for relativistic case :-

$$L = -m_0c^2\sqrt{1-\frac{\vec{v}^2}{c^2}}$$

$$L = -m_0c^2\sqrt{1-\frac{v_i^2}{c^2}} \qquad \left\{\because \vec{a} \to a_i\right\}$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial v_i}$$

$$p_i = \frac{-m_0\cancel{c^2}}{\cancel{2}\sqrt{1-\frac{v_i^2}{c^2}}}\left(-\frac{\cancel{2}v_i}{\cancel{c^2}}\right)$$

$$p_i = \frac{m_0 v_i}{\sqrt{1-\frac{v_i^2}{c^2}}}$$

Now squaring both side

$$p_i^2\left(1-\frac{v_i^2}{c^2}\right) = m_0^2 v_i^2$$

$$p_i^2 = v_i^2\left[m_0^2 + \frac{p_i^2}{c^2}\right]$$

$$v_i^2 = \frac{p_i^2 c^2}{m_0^2 c^2 + p_i^2}$$

$$H = p_i v_i - L$$

$$= p_i v_i + m_0 c^2\sqrt{1-\frac{v_i^2}{c^2}}$$

$$= \frac{p_i p_i c}{\sqrt{m_0^2 c^2 + p_i^2}} + m_0 c^2\sqrt{1-\frac{p_i^2}{m_0^2 c^2 + p_i^2}}$$

$$= \frac{p_i^2 c}{\sqrt{m_0^2 c^2 + p_i^2}} + \frac{m_0 c^2\, m_0 c}{\sqrt{m_0^2 c^2 + p_i^2}}$$

$$= \frac{c\,(p_i^2 + m_0^2 c^2)}{\sqrt{p_i^2 + m_0^2 c^2}} = c\sqrt{p_i^2 + m_0^2 c^2}$$

$$= \sqrt{p_i^2 c^2 + m_0^2 c^4}$$

So $\boxed{H = \sqrt{p^2 c^2 + m_0^2 c^4}}$

Net - Lewis ...

Q. If $L = -\sqrt{1-\dot{q}^2} - V(q)$ find $H = ?$

Sol^n

$$\boxed{H = \sqrt{p_q^2 + 1} + V(q)}$$

Standard form. {here $m_0 = 1$, $c = 1$

* Question on $H \longrightarrow L$:-

Q. $H = \frac{p^2}{2m} + \vec{a}\cdot\vec{p}$ find Lagrangian ?

Sol^n

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \Rightarrow \because H = \frac{p_i^2}{2m} + a_i p_i$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = v_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m} + a_i$$

$$p_i = (v_i - a_i)m$$

$$L = p_i\dot{q}_i - H = p_i\dot{q}_i - \frac{p_i^2}{2m} - a_i p_i$$

put p_i

$$L = (v_i - a_i)m v_i - \frac{m}{2}(v_i - a_i)^2 - a_i m(v_i - a_i)$$

$$= m(v_i - a_i)(v_i - a_i) - \frac{m}{2}(v_i - a_i)^2$$

$$= m(v_i - a_i)^2 - \frac{m}{2}(v_i - a_i)^2$$

$$= \frac{1}{2}m(v_i - a_i)^2$$

So $\boxed{L = \frac{1}{2}m(\vec{v} - \vec{a})^2}$ Ans

Q. Hamiltonian of the system is $H = ap_r^2 + \frac{bp_\theta^2}{r^2} + c\cos\theta$

Where a, b, c are constants find lagrangian?

Sol^n Here two co-ordinates are used which is r and θ. So velocity corresponding to r is $\dot{r}$ and corresponding to θ is $\dot{\theta}$.

$$q_i = r, \theta$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = 2ap_r$$

$$\boxed{p_r = \frac{\dot{r}}{2a}}$$

Now

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{2bp_\theta}{r^2} \Rightarrow \boxed{p_\theta = \frac{r^2\dot{\theta}}{2b}}$$

$$L = p_i \dot{q}_i - H$$

$$= p_r\dot{r} + p_\theta\dot{\theta} - ap_r^2 - \frac{bp_\theta^2}{r^2} - c\cos\theta$$

Put p_r and p_θ

$$L = \frac{\dot{r}^2}{2a} + \frac{r^2\dot{\theta}^2}{2b} - \frac{\dot{r}^2}{4a} - \frac{r^2\dot{\theta}^2}{4b} - c\cos\theta$$

$$\boxed{L = \frac{\dot{r}^2}{4a} + \frac{r^2\dot{\theta}^2}{4b} - c\cos\theta}$$

* Lagrangian and Hamiltonian of a particle in different co-ordinate system:-

(I) Plane Polar (r, θ) :-

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \text{P.E. term}$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \text{P.E. term.}$$

(II) Spherical Polar (r, θ, ϕ) :-

$$L = \frac{1}{2} m (1 \cdot \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2) - \text{P.E. term}$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2\theta} + \text{P.E.}$$

(III) Cylindrical Coordinate (s, ϕ, z) or (r, ϕ, z) :-

$$L = \frac{1}{2} m (\dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2) - \text{P.E. term.}$$

$$H = \frac{p_s^2}{2m} + \frac{p_\phi^2}{2ms^2} + \frac{p_z^2}{2m}$$

Q. If $H = a p_\theta^2 + b \cos\theta$ write eqn of motion of system (2nd order differential eqn).

Soln Write Hamilton's Equation.

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = 2ap_\theta$$

$$\dot{p}_\theta = \frac{-\partial H}{\partial \theta} = b\sin\theta$$

$$\boxed{\begin{aligned}\dot{\theta} &= 2ap_\theta \\ \dot{p}_\theta &= b\sin\theta\end{aligned}} \longrightarrow \text{first order differential Equation.}$$

To get second order diff. eq^n in θ put $\dot{p}_\theta$ in Ist Equation.

diff. w.r.t. time.

first eq^n - $\ddot{\theta} = 2a\dot{p}_\theta$

$$\ddot{\theta} = 2ab\sin\theta$$

$$\boxed{\ddot{\theta} - 2ab\sin\theta = 0} \longleftarrow \text{second order differential eq}^n \text{ in } \theta.$$

Q. If $H = \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta)$ find second order diff. eq^n in θ.

Sol^n

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{2p_\theta}{2ml^2}$$

$$\dot{p}_\theta = \frac{-\partial H}{\partial \theta} = \cancel{+}\, mgl\sin\theta$$

$$\ddot{\theta} = \frac{\dot{p}_\theta}{ml^2} = -\frac{mgl\sin\theta}{ml^2}$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

* Poisson's Bracket :- {P.B.}

If A and B are two dynamical variables -

i.e. $A = A(p_i, q_i, t)$, $B = B(p_i, q_i, t)$

possion Bracket of A with B is defined as-

$$\boxed{\{A, B\}_{q_i, p_i} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}}$$

Summation over i.

$\{q_i, p_i\} = 1$, $\{x, p_x\} = 1$

$\{q_i, p_i\} = \delta_{ij}$, $\{x, p_y\} = 0$

← Fundamental P.B.

Properties :-

* $\{A, B\} = -\{B, A\}$
* $\{A, B+C\} = \{A, B\} + \{A, C\}$
* $\{A, BC\} = \{A, B\}C + B\{A, C\}$ → Leibnitz Identity
* $\{A, \{B, C\}\} + \{C, \{A, B\}\} + \{B, \{C, A\}\} = 0$ ← Jacobi Identity

* Poisson's Bracket with angular momentum:-

$[x, L_y] = z$, $[y, L_z] = x$

$[p_x, L_z] = -p_y$

$[p_z, L_x] = p_y$

$[L_x, L_y] = L_z$

$[p_y, L_y] = 0$

$[q, L_q] = 0$

If we replace $i\hbar \to 1$, then Commutator bracket changes into P.B. and vice-versa.
Here in P.B. we can use { } or [ ]

* General Relation :-

$$[A_i, L_j] = \varepsilon_{ijk} A_k$$

$$[A_i, L_j] = \varepsilon_{ijl} A_l$$

Here l is repeated index so we have summation over it, so we can change variable.

Levicivita Tensor.

* A and B are constant of motion (Conserved) then their possion bracket is also constant of motion.

Q. If $A = ap_1 + bq_2$, $B = cq_1 + dp_2$ find the value of P.B. $\{A, B\}$.

Note :- Here A and B depends upon position and momentum co-ordinate so these two are dynamical variable. formula $\Rightarrow \{A, B\}_{q_i, p_i} = \left[\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right]$

$$\{A, B\}_{q_i, p_i} = \{\underbrace{ap_1 + bq_2}_{A}, \underbrace{cq_1 + dp_2}_{B}\}_{q_1, p_1}$$

$$+ \{\underbrace{ap_1 + bq_2}_{A}, \underbrace{cq_1 + dp_2}_{B}\}_{q_2, p_2}$$

$$= (0 - a.c) + (bd - 0)$$

$$= bd - ac \quad \text{Ans}$$

Q. Explicit calculation of $[x, L_y]$

Sol^n

$$[x, zp_x - xp_z] = [\underbrace{x}_{A}, \underbrace{zp_x - xp_z}_{B}]_{x,p_x} + [\underbrace{x}_{A}, \underbrace{zp_x - x}_{B}]_{z}$$

$$= 1.z + 0 + 0 + 0$$

$$= z \quad \underline{\text{Ans}}$$

B.A.

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Sol^n

$$\{x_i, p_j\} = \delta_{ij}$$

$$C_1 = x_2 p_3 + x_3 p_2$$

$$C_2 = x_1 p_2 - x_2 p_1$$

$$C_3 = x_1 p_3 + x_3 p_1 = \{C_1, C_2\}$$

(a) $\{C_1, C_3\} = C_1$ & $\{C_1, C_3\} = C_2$

(b) $\{C_2, C_3\} = -C_1$ & $\{C_3, C_1\} = -C_2$

$$\{C_2, C_3\} = \{x_1 p_2 - x_2 p_1, x_1 p_3 + x_3 p_1\}$$

$$= \{\quad\}_{x_1 p_1} + \{\quad\}_{x_2 p_2} + \{\quad\}_{x_3 p_3}$$

$$= p_2 x_3 + x_2 p_3 - p_1 \times 0 - x_1 \times 0 + 0 - 0$$

$$\{C_2, C_3\} = C_1$$

B.A. Note.

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Sol^n $L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$

P.B $\{\theta, \dot{\theta}\} = ?$ $\because \{\theta, p_\theta\} = 1$ {stand.}

Convert $\dot{\theta}$ into p_θ.

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\dot{\theta} = \frac{p_\theta}{ml^2}$$

$$\{\theta, \dot{\theta}\} = \{\theta, \frac{p_\theta}{ml^2}\} = \frac{1}{ml^2}\{\theta, p_\theta\} = \frac{1}{ml^2} \times 1$$

$$\boxed{\{\theta, \dot{\theta}\} = \frac{1}{ml^2}}$$ Ans

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Q.9

Sol^n

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{qB}{2}(x\dot{y} - y\dot{x})$$

$$[\dot{x}, \dot{y}] = ?$$

$$\because \{\dot{x}, p_{\dot{x}}\} = 1$$

Express $\dot{x}$ and $\dot{y}$ in terms of co-ordinate and momentum.

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{qB}{2}y$$

$$\dot{x} = \frac{\left(p_x + \frac{qB}{2}y\right)}{m}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{qB}{2}x$$

$$\dot{y} = \frac{p_y - \frac{qB}{2}x}{m}$$

$$[\dot{x}, \dot{y}] = \left[\frac{p_x + \frac{qB}{2}y}{m}, \frac{p_y - \frac{qB}{2}x}{m}\right]$$

$$= \left[\frac{p_x + \frac{qB}{2}y}{m}, \frac{p_y - \frac{qB}{2}x}{m}\right]_{x, p_x} + \left[\frac{p_x + \frac{qB}{2}y}{m}, \frac{p_y - \frac{qB}{2}x}{m}\right]_{y p_y}$$

Q. Evaluate $\{(\vec{a}\cdot\vec{r}), \vec{p}\}$ Where $\vec{a}$ is constant

Sol^n

$$\because \{(\vec{a}\cdot\vec{r}), \vec{p}\} = \{a_i r_i, p_j\}$$

$$= a_i \{r_i, p_j\}$$

$$= a_i \delta_{ij} = a_j \times 1$$

$$= a_j$$

$$= \vec{a}. \quad \underline{\text{An}}$$

Q. $\{(\vec{a}\cdot\vec{r}), \vec{r}\}$ evaluate ?

Sol^n $\{(\vec{a}\cdot\vec{r}), \vec{r}\} = 0$ $\quad$ ($\because$ Here no momentum term ~~absent~~)

Q. Evaluate $\{(\vec{a}\cdot\vec{r})^2, \vec{p}\}$

Sol^n $\{(\vec{a}\cdot\vec{r})^2, \vec{p}\} = \{\underline{(\vec{a}\cdot\vec{r})}\,\underline{(\vec{a}\cdot\vec{r})}, \underline{\vec{p}}\}$

$\because \{AB, C\} = A\{B, C\} + B\{A, C\}$

$\therefore \{(\vec{a}\cdot\vec{r})^2, \vec{p}\} = (\vec{a}\cdot\vec{r})\{(\vec{a}\cdot\vec{r}), \vec{p}\} + (\vec{a}\cdot\vec{r})\{(\vec{a}\cdot\vec{r}), \vec{p}\}$

$= 2(\vec{a}\cdot\vec{r})\{(\vec{a}\cdot\vec{r}), \vec{p}\}$

$= 2(\vec{a}\cdot\vec{r})\vec{a}$ $\quad \{\because \{(\vec{a}\cdot\vec{r}), \vec{p}\} = \vec{a}\}$

Ans

Second Method :-

$\{(\vec{a}\cdot\vec{r})^2, \vec{p}\} = \{a_i^2 r_i^2, p_j\}$

$= a_i^2 \{r_i^2, p_j\}_{r_i, p_i}$

$= a_i^2 \left[2r_i \frac{\partial p_j}{\partial p_i} - 0\right]$

$= a_i^2\, 2r_i\, \delta_{ij}$

$= 2a_j^2 r_j$ $\quad \{\because \delta_{ij} = 1 \text{ when } i=j\}$

$= 2a_j(a_j r_j)$

$= 2a_j(\vec{a}\cdot\vec{r})$

$= 2\vec{a}(\vec{a}\cdot\vec{r})$ Ans

Q. Evaluate $[\vec{a}.\vec{r}, \vec{L}]$

angular momentum not Lagrangian becoz Lagrangian is scalar quantity

Sol^n $[a_i r_i, L_j] = a_i [r_j, L_j]$

$= a_i \varepsilon_{ijk} r_k$

$= \varepsilon_{ijk} a_i r_k$

$= -\varepsilon_{jik} a_i r_k$

$= -(\vec{a} \times \vec{r})_j \quad \left\{ \because (\vec{a} \times \vec{b})_i \; \varepsilon_{ijk} a_i b_k \right\}$

$= (\vec{r} \times \vec{a})_j$

$= (\vec{r} \times \vec{a})$ Ans

Q. Evaluate $[\vec{a}.\vec{L}, \vec{b}.\vec{L}]$

$\Rightarrow [a_i L_i, b_j L_j]$

$\Rightarrow a_i b_j [L_i, L_j]$

$= a_i b_j \varepsilon_{ijk} L_K$

$= a_i \varepsilon_{ijk} b_j L_k$

$= a_i (\vec{b} \times \vec{L})_i$

$= \vec{a}.[\vec{b} \times \vec{L}] = [\vec{a} \times \vec{b}).\vec{L}$

Q. Evaluate $[x, L_y] = z$, $[L_x, y] = ?$

$$[yp_z - zp_y, y] = [\quad]_{y,p_y} + [\quad]_{z,}$$

$$= p_z \times 0 + z \times 1 + (-p_y) \times 0 - y = 0$$

$$= z$$

$[p_x, L_y] = p_z$

$[L_x, L_y] = p_z$

$[L_x, y] = z$

Q. Evaluate $[\vec{r}, \vec{a}.\vec{L}]$ where $\vec{a}$ = Constant vector.

$$[r_i, a_j L_j] = a_j [r_i, L_j]$$

$$= a_j \varepsilon_{ijk} r_k$$

$$= \varepsilon_{ijk} a_j r_k$$

$$= (\vec{a} \times \vec{r})_i = \boxed{\vec{a} \times \vec{r}} \text{ Ans}$$

Create

Q. Evaluate $\{\vec{a}.\vec{r}, \vec{b}.\vec{p}\}$

$$= \{a_i r_i, b_j p_j\}$$

$$= a_i b_j \{r_i, p_j\}$$

$$= a_i b_j \delta_{ij}$$

$$= a_j b_j \times 1 \quad = \vec{a}.\vec{b} \text{ Ans}$$

Q. Evaluate $[|\vec{r}|, |\vec{p}|]$

Sol^n $\because r = \sqrt{x^2+y^2+z^2}$

So $\left[\sqrt{x^2+y^2+z^2}, \sqrt{p_x^2+p_y^2+p_z^2}\right]$

$= [\quad]_{xp_x} + [\quad]_{yp_y} + [\quad]_{zp_z}$

$= \left(\frac{x}{|\vec{r}|}\cdot\frac{p_x}{|\vec{p}|} + 0\right) + \left(\frac{y}{|\vec{r}|}\frac{p_y}{|\vec{p}|} + 0\right) + \left(\frac{z}{|\vec{r}|}\frac{p_z}{|\vec{p}|} + 0\right)$

$= \frac{xp_x + yp_y + zp_z}{|\vec{r}||\vec{p}|} = \frac{\vec{r}\cdot\vec{p}}{|\vec{r}||\vec{p}|}$

$= \frac{\vec{r}}{|\vec{r}|}\cdot\frac{\vec{p}}{|\vec{p}|} = \boxed{\hat{r}\cdot\hat{p}}$ Ans

Q. If $x' = x\cos\theta - p_x\sin\theta$, $p_x' = x\sin\theta + p_x\cos\theta$.
evaluate P.B. $\{x', p_x'\}_{x, p_x}$

Sol^n

$\{x\cos\theta - p_x\sin\theta, x\sin\theta + p_x\cos\theta\}_{x, p_x}$

$= \cos\theta \times \cos\theta + \sin\theta \times \sin\theta$

$= \cos^2\theta + \sin^2\theta$

$= 1.$

* Poission's Equation of Motion :-

$$\frac{dA}{dt} = [A, H] + \frac{\partial A}{\partial t}$$

If $A = H$

$$\frac{dH}{dt} = [H, H] + \frac{\partial H}{\partial t}$$

$$\boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t}}$$

If H does not depends on t explicitly.

$$\frac{\partial H}{\partial t} = 0$$

$$\frac{dH}{dt} = 0 \Rightarrow \boxed{H = \text{Constant or Conserved.}}$$

If H does not depends on time explicitly then H is conserved.

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Net-2012
Dec.

Q. A system is governed by the Hamiltonian $H = \frac{1}{2}(p_x - ay)^2 + \frac{1}{2}(p_y - bx)^2$ where a and b are constants and p_x, p_y are momenta conjugate of to x and y respectively. For what values of a and b will the quantities $(p_x - 3y)$ and $(p_y + 2x)$ be conserved.

Sol^n $H = \frac{1}{2}(p_x - ay)^2 + \frac{1}{2}(p_y - bx)^2$

$p_x - 3y$, $p_y + 2x$ conserved.

Let $A = p_x - 3y$ it does not depends on time explicitely

* Imp. So $\frac{\partial A}{\partial t} = 0$ so it is conserved

So $\frac{dA}{dt} = 0$

$\therefore [A, H] = 0$

$\therefore$ $B = p_y + \cancel{2x}$ it is also it does not depends on time explicitely

So $\frac{\partial B}{\partial t} = 0$ So it is also conserved

So $\frac{dB}{dt} = 0$

$\therefore [B, H] = 0$

$$\Bigg[\underbrace{p_x - 3y}_{A}, \underbrace{\frac{(p_x - ay)^2}{2} + \frac{(p_y - b\cancel{x})^2}{2}}_{B}\Bigg] = 0$$

$$\Rightarrow [\quad]_{x, p_x} + [\quad]_{y, p_y} = 0$$

$$0 - 1.(p_y - bx)(-b) + (-3)(p_y - bx) - 0 = 0$$

$$\Rightarrow (p_y - bx)(b - 3) = 0$$

$$b - 3 = 0$$

$$\boxed{b = 3}$$

$$[B,H] = [\underbrace{p_y + 2x}_{A},\ \underbrace{\frac{(p_x - ay)^2}{2} + \frac{(p_y - bx)^2}{2}}_{B}] = 0$$

$$[B,H] = [\underbrace{(p_y+2x)}_{A},\ \underbrace{\frac{(p_x-ay)^2}{2} + \frac{(p_y-bx)^2}{2}}_{B}]_{x,p_x} + [\underbrace{(p_y+2x)}_{A},\ \underbrace{\frac{(p_x-ay)^2}{2} + \frac{(p_y-bx)^2}{2}}_{B}]_{y,p_y}$$

$$\left[\frac{\partial A}{\partial x}\cdot\frac{\partial B}{\partial p_x} - \frac{\partial A}{\partial p_x}\cdot\frac{\partial B}{\partial x}\right] \qquad \left[\frac{\partial A}{\partial y}\cdot\frac{\partial B}{\partial p_y} - \frac{\partial A}{\partial p_y}\cdot\frac{\partial B}{\partial y}\right]$$

$$= \left[\{2\cdot(p_x - ay)\} - \{0\}\right] + \left[\cancel{0 + 1\cdot(p_y - b)}\right]$$

$$[0 + 1\cdot(p_x - ay)(-a)$$

$$\cancel{= 2(p_x - ay) - (p_y - b)}$$

$$2(p_x - ay) + (p_x - ay)(+a)$$

$$(p_x - ay)(2+a) = 0$$

$$a = -2$$ Ans

(*) "Conservation Laws :-

(1) If L or H is translationally invariant then linear momentum is conserved in that direction.

$$p_x = \frac{\partial L}{\partial \dot{x}} \quad , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}(p_x) - \frac{\partial L}{\partial x} = 0$$

$$\dot{p}_x = \frac{\partial L}{\partial x}$$

If L does not explicitly depend on x

$$\frac{\partial L}{\partial x} = 0$$

$$\dot{p}_x = 0 \Rightarrow p_x = \text{Constant}$$

Hamiltonian eq^n of motion -

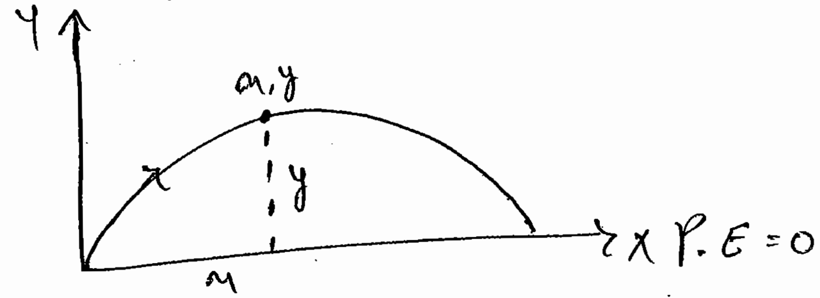
$$\dot{p}_x = -\frac{\partial H}{\partial x}$$

If H is not explicit function of x

$$\frac{\partial H}{\partial x} = 0$$

$$\dot{p}_x = 0 \quad , \quad p_x = \text{Constant}$$

Ex-



$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

x is cyclic

Translation $x \longrightarrow x + \alpha$, then $L \longrightarrow L$

Here L is invariant w.r.to translation x-direcn

$\therefore$ p_x is conserved.

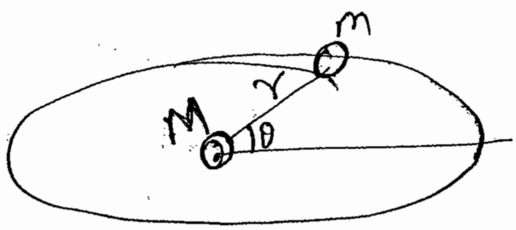
* Homoginity of space leads to conservation of momentum.

Second Theorem:-

If L or H is Rotationally invariant, Angular momentum is conserved in that direction.

$$\theta \longrightarrow \theta + \alpha, \quad L \rightarrow L$$

Ex - Sun - Planet System -



$$L = \frac{1}{2} m (\dot{r}^2 + r\dot{\theta}^2) + \frac{GMm}{r}$$

$\theta \rightarrow$ is cyclic, $\frac{\partial L}{\partial \theta} = 0$

$$\dot{p}_\theta = \frac{\partial L}{\partial \theta}$$

$$\dot{p}_\theta = 0$$

$$\boxed{p_\theta = \text{Constant}}$$

So $$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{GMm}{r}$$

θ is cyclic $\frac{\partial H}{\partial \theta} = 0$

Hamilton's equation -

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

$$\boxed{p_\theta = \text{Conserved}}$$

Angular momentum is Conserved.

$$\boxed{p_\theta = L_z}$$

* Isotropy of space leads to Conservation of angular momentum.

* Third Theorem :-

If L or H does not explicitly depends on time then Hamiltonian is conserved.

$$\frac{dA}{dt} = [A, H] + \frac{\partial A}{\partial t}$$

$$A = H$$

$$\boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t}} \rightarrow 0$$

$$\frac{dH}{dt} = 0$$

$$\boxed{H = \text{Constant}}$$

(a) If potential energy does not depends on velocity then H = total energy.

(b) If potential energy depends on velocity then

$$\boxed{H \neq \text{total energy}}$$

Imp.

* If L or H does not depends on t explicitly and potential does not depends on velocity then total energy is conserved.

** If potential depends on velocity then H is conserved but total is not conserved.

*** Homogeneity of time leads to conservation of Hamiltonian (Energy).

Q. $L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} K(x^2 + y^2)$ Which of the following is conserved.

p_x, p_y, L_z, E = energy

Sol^n x and y are not cyclic so p_x and p_y are not conserved.

To know about L_z write L in plane polar co-ordi

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2} K r^2$$ here θ is not com

So θ is cyclic.

$\therefore p_\theta$ = Constant

$\therefore L_z$ = Constant

L does not explicitly depends on t.

So H = Conserved.

∵ Potential does not depend on velocity

∴ $H = E$

So $\boxed{E = \text{Conserved}}$

* How to identify the K.E. term and P.E. term in L.

⇒ Say $L = \underbrace{\frac{1}{2} m (\dot{x}^2 + \dot{y}^2)}_{\text{K.E. term}} - \underbrace{\frac{1}{2} K (x^2+y^2) + \omega (x\dot{y} - y\dot{x})}_{\text{P.E. term.}}$

∵ K.E. term is always quadratic in velocity.
like $\dot{x}^2$, $\dot{y}^2$ or $(\dot{x}\dot{y})$ etc.
in $\dot{q}_i$ i.e. $\dot{q}_i^2$ or $\dot{q}_i \dot{q}_j$

$H = \text{Constant}$

$H \neq E$

∴ E is not conserved.

A-11
Q.11 Lagrangian of a system is $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + qEy - \frac{qB}{c} y\dot{x}$
Which of the following is not correct.

Solⁿ $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + qEy - \frac{qB}{c} y\dot{x}$

(a) $m\dot{x} - \frac{qBy}{c} = \text{Const.}$

(b) $m\dot{z} = \text{Const.}$

× (c) $m\dot{y} + \frac{qBx}{c} + qEt = \text{Const.}$ ✓

(d) $m\dot{y} + \frac{qBy}{c} - qEt = \text{Const.}$

$\therefore$ x and z are constant.

$p_x = \text{Constant}$, $p_z = \text{Constant}$.

$$p_x = \frac{\partial L}{\partial \dot{x}}$$

$$p_x = m\dot{x} - \frac{qB}{c}y = \text{Constant}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} = \text{Constant}$$

~~If $m\dot{y}$~~ say $A = m\dot{y} + \frac{qBx}{c} + qEt$

If $A = \text{Constant}$

then $\frac{dA}{dt} = 0$

$$\frac{dA}{dt} = m\ddot{y} + \frac{qB\dot{x}}{c} + qE \quad \text{—— (I)}$$

write Lagrangian's eqⁿ in y –

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$m\ddot{y} - qE + \frac{qB}{c}\dot{x} = 0$$

$$m\ddot{y} + \frac{qB}{c}\dot{x} = qE \quad \text{—— (II)}$$

So from (I)

$$\therefore \frac{dA}{dt} = 2qE \neq 0$$

So (c) is wrong. Ans

Q. 98

Sol^n

$$L = \underbrace{\mu(\dot{x}_1^2 + \dot{x}_2^2)}_{K.E.} - \underbrace{\gamma(x_1 + x_2)}_{P.E.}$$

$\because$ L is not explicitly depend on time so

H = Conserved

$\because$ in P.E. part is independent of velocity

So H = total Energy

$\therefore$ H = Conserved

$\therefore$ $\boxed{E = \text{Conserved}}$

Here x_1 & x_2 are not cyclic

$\therefore$ p_1, p_2 are not conserved,

$$p_1 = \frac{\partial L}{\partial \dot{x}_1} = 2\mu \dot{x}_1$$

$$p_2 = \frac{\partial L}{\partial \dot{x}_2} = 2\mu \dot{x}_2$$

$\dot{x}_1$ & $\dot{x}_2$ are not constant. (b) E & $\dot{x}_1 - \dot{x}_2$ is conserved

$\because$ (a) E & $\dot{x}_1 + \dot{x}_2$ is conserved

let's check say $A = \dot{x}_1 - \dot{x}_2$

$$\frac{dA}{dt} = \ddot{x}_1 - \ddot{x}_2 \quad \text{---} (*)$$

Say $B = \dot{x}_1 + \dot{x}_2$

$$\frac{dB}{dt} = \ddot{x}_1 + \ddot{x}_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$2M\ddot{x}_1 + \gamma = 0$$

$$\boxed{\ddot{x}_1 = \frac{-\gamma}{2M}}$$

$||^{ly}$

$$\boxed{\ddot{x}_2 = \frac{-\gamma}{2M}}$$

Put in $(*)$ get $\frac{dA}{dt} = 0$

So option (b) is correct.

A-11 Net

Q. 19

Sol^n

$$H = \underbrace{\frac{p_\theta^2}{2ml^2}}_{T} + \underbrace{mgl(1 - \cos\theta)}_{V}$$

If L = Lagrangian & $\frac{dL}{dt} = ?$

$$\frac{dA}{dt} = [A, H] + \frac{\partial A}{\partial t}$$

$$\frac{dL}{dt} = [L, H] + \frac{\partial L}{\partial t} = 0$$

becoz L is does not depend on time explicitely

$$\boxed{\frac{dL}{dt} = [\underline{L}, \underline{H}]}$$

$L = p_i \dot{q}_i - H$

$L = p_\theta \dot{\theta} - H$

$\because \quad L = T - V$

$$L = \frac{p_\theta^2}{2ml^2} - mgl(1-\cos\theta)$$

$$\frac{dL}{dt} = [L, H] = \Big[\underbrace{\frac{p_\theta^2}{2ml^2} - mgl(1-\cos\theta)}_{A}, \underbrace{\frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta)}_{B}\Big]$$

$$= \left[\frac{p_\theta^2}{2ml^2} - mgl(1-\cos\theta), \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta)\right]_{\theta, p_\theta}$$

$$= -\cancel{m}gl\sin\theta \times \frac{p_\theta}{\cancel{m}l^2} - \frac{p_\theta}{\cancel{m}l^2} \cdot \cancel{m}gl\sin\theta$$

$$\boxed{\frac{dL}{dt} = -\frac{2g\,p_\theta \sin\theta}{l}}$$

* Phase Space Dynamics :-

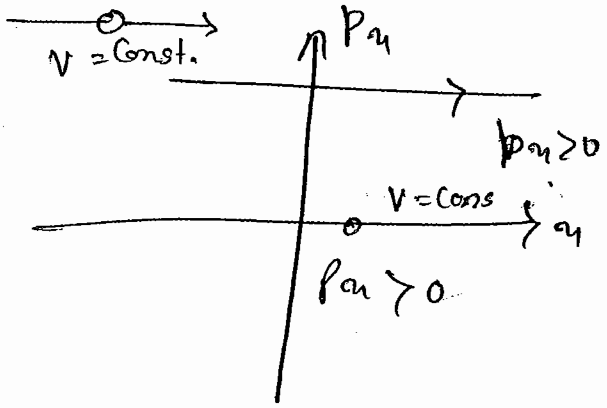
How momentum varies with coordinate (graph b/w p_i, q_i)

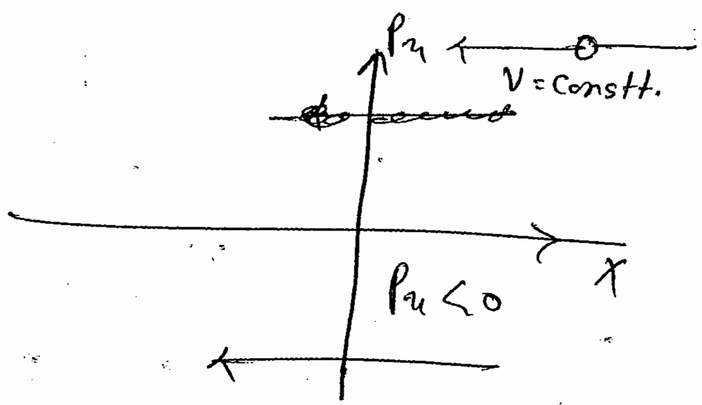
Phase space : It consists of co-ordinates and momenta.

for a system having f-D.O.F. phase space is a 2f dimensional space (f-coordinate + f momenta).

How to draw phase space line (phase space trajectory) [for one dimensional problem]

* If particle is moving towards +ve x direc^n then $p_x > 0$ (+ve)
* If particle is moving towards -ve x direc^n $p_x < 0$ (-
* If $p_x > 0$ then arrow should be towards +ve X
* If $p_x < 0$ then arrow should be towards -ve X.





* Cauchy Lipschitz Condition :-

The two phase space trajectory can not intersect each other.

* for Conservative System (Energy = Constant):-

If $V(x)$ [P.E.] is increasing then K.E. must decrease [p_x must decrease] and vice-versa.

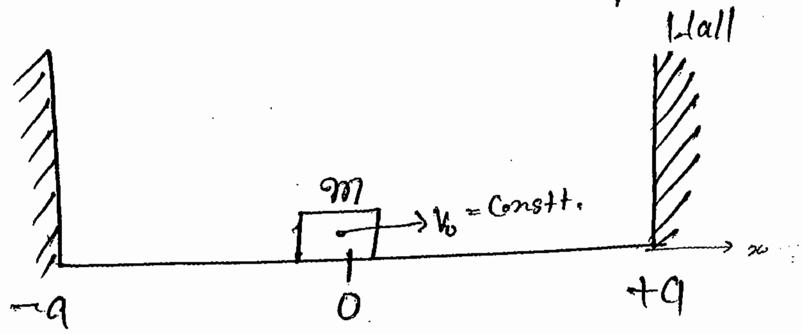
Therefore we first draw $V(x)$ and then using if we draw phase space lines for conservative system.

* Types of qustions :-

① Potential Energy is Given

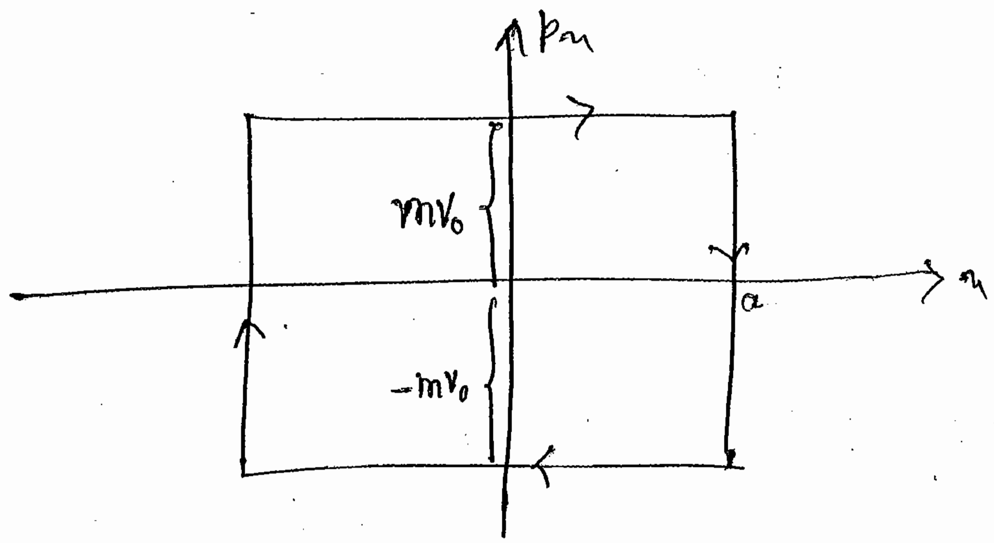
② Some information about dynamics is given

Q.

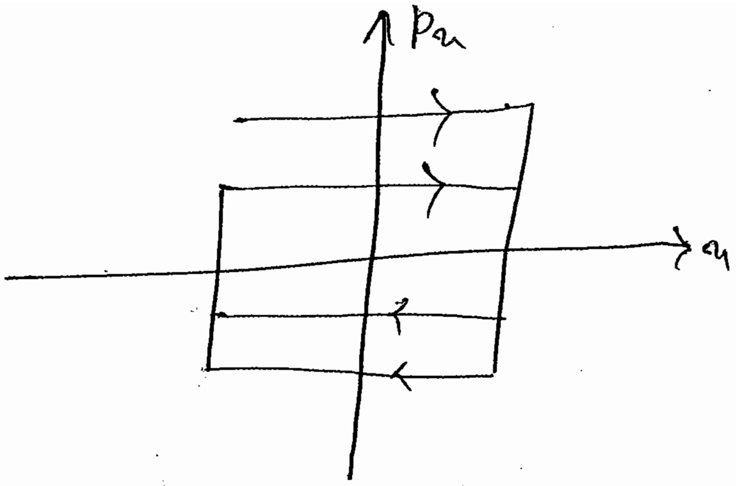


If collisions are elastic draw phase space line.

Sol^n

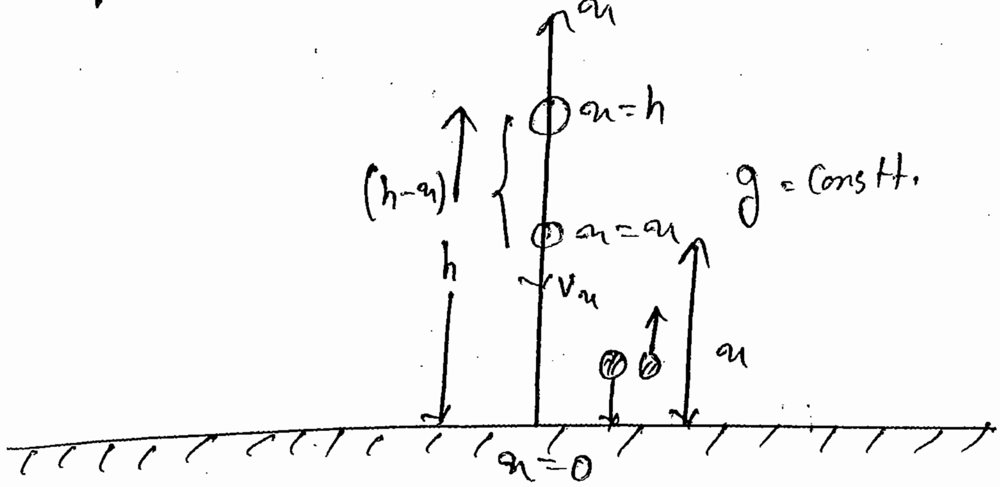


(ii) If Collisions are inelastic :-



P.

Q. A ball dropped from some height on horizontal surface.



$$v^2 = u^2 + 2as$$

$$v_x^2 = 0 + 2g(h-x)$$

$$v_x^2 = 2g(h-x)$$

$$\boxed{\frac{p_x^2}{m^2} = 2g(h-x)} \leftarrow \text{parabola}$$

$$p_x^2 = 2g(h-x)m^2$$

put $h - x = x'$

$$\boxed{p_x^2 = 2gm^2x'}$$

$$v_x^2 = 2g(h-x)$$

$$v_x = \frac{p_x}{m}$$

$$p_x^2 = 2gm^2(h-x)$$

$$p_x \frac{dp_x}{dx} = -2gm^2$$

$$\frac{dp_x}{dx} = -\frac{2gm^2}{p_x}$$

$$\frac{d^2p_x}{dx^2} = \frac{2gm^2}{p_x^2} \cdot$$

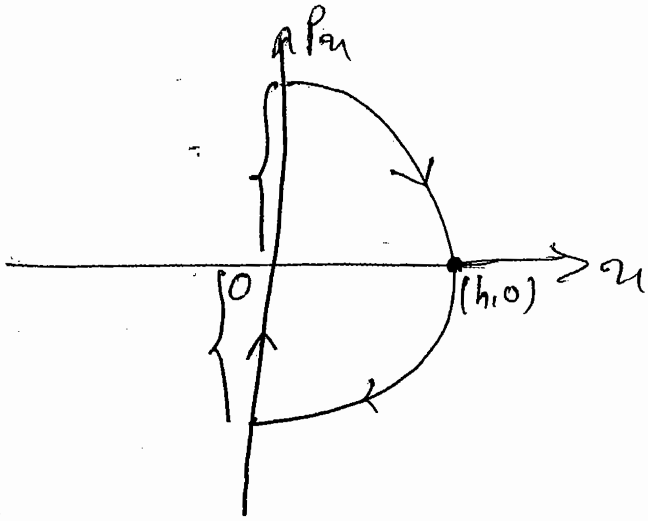
$$\frac{d^2p_x}{dx^2} = -\frac{4g^2m^4}{p_x^3}$$

$p_x \not< 0$ so $\frac{d^2p_x}{dx^2} > 0$

So trajectory is +ve.

Rule :-

$$E = \frac{p_x^2}{2m} + mgx$$



Elastic

$$p_x \longrightarrow -p_x$$

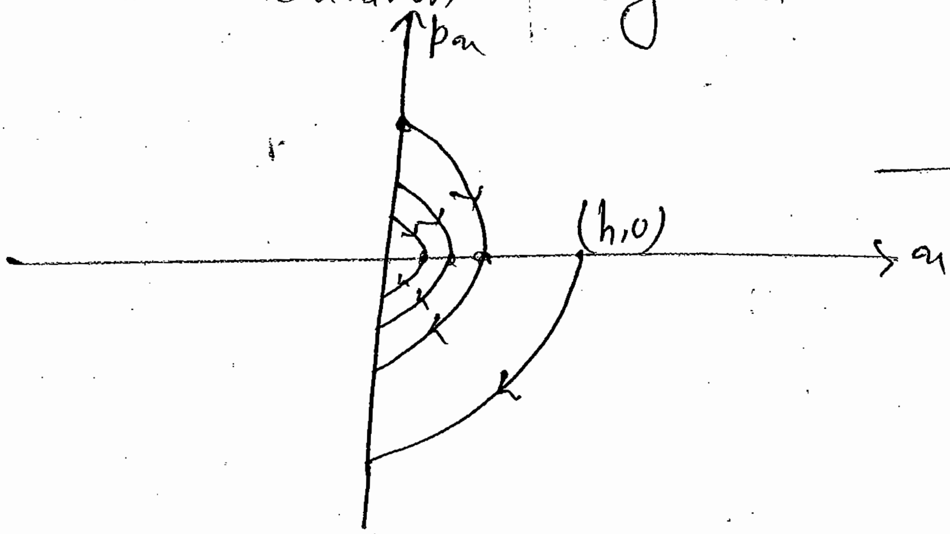
E does not change then symmetry about x-axis.

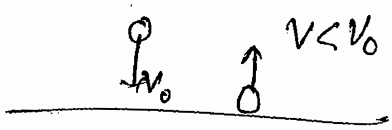
When $x \longrightarrow -x$

E does not change then symmetry about p_x axis.

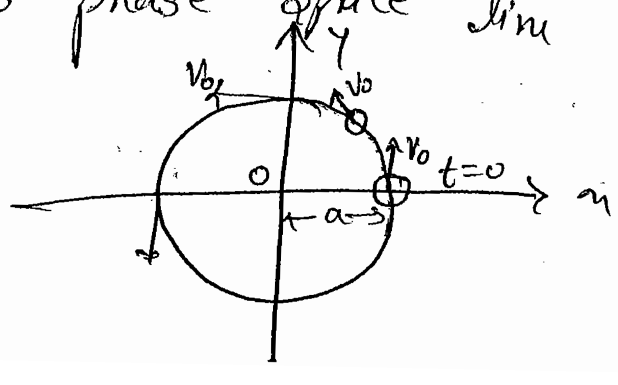
(ii) If Collision is inelastic :-

Inelastic collision with ground.



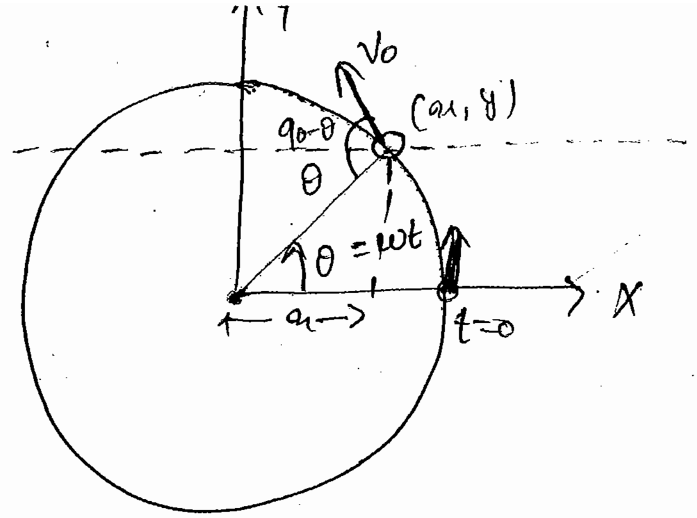


Q. A particle is moving in a circle with constant speed. draw phase space line in $x - p_x$ space.



$\therefore \quad \omega = \frac{d\theta}{dt}$

$$\boxed{\omega = \frac{v_0}{a}}$$



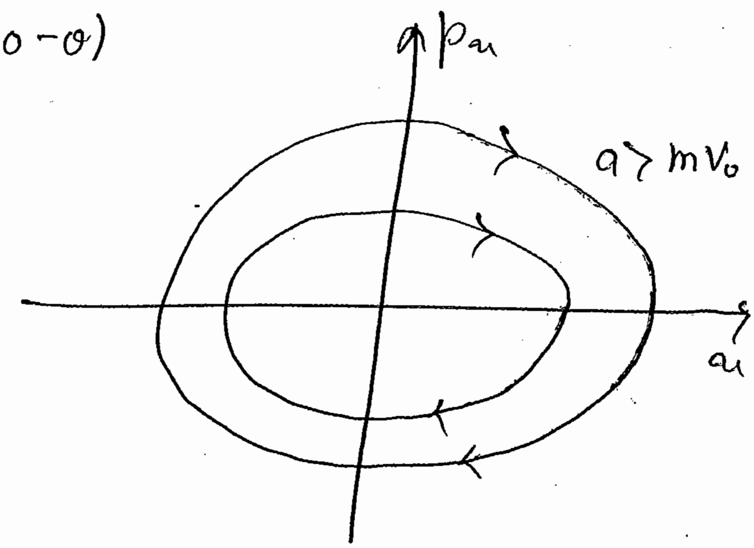
$x = a\cos\theta$

$x = \cos\omega t$

$V_x = -V_0 \cos(90-\theta)$

$\frac{p_x}{m} = -V_0 \sin\theta$

$\frac{p_x}{mV_0} = -\sin\omega t$

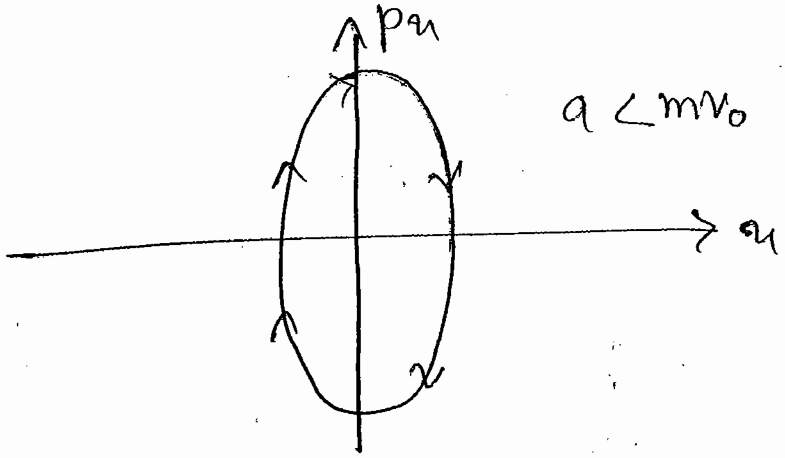


$\frac{x}{a} = \cos\omega t$

$\frac{p_x}{mV_0} = -\sin\omega t$

Square and add

$$\boxed{\frac{x^2}{a^2} + \frac{p_x^2}{(mV_0)^2} = 1}$$ eq^n of ellipse.



* Potential Based Questions:-

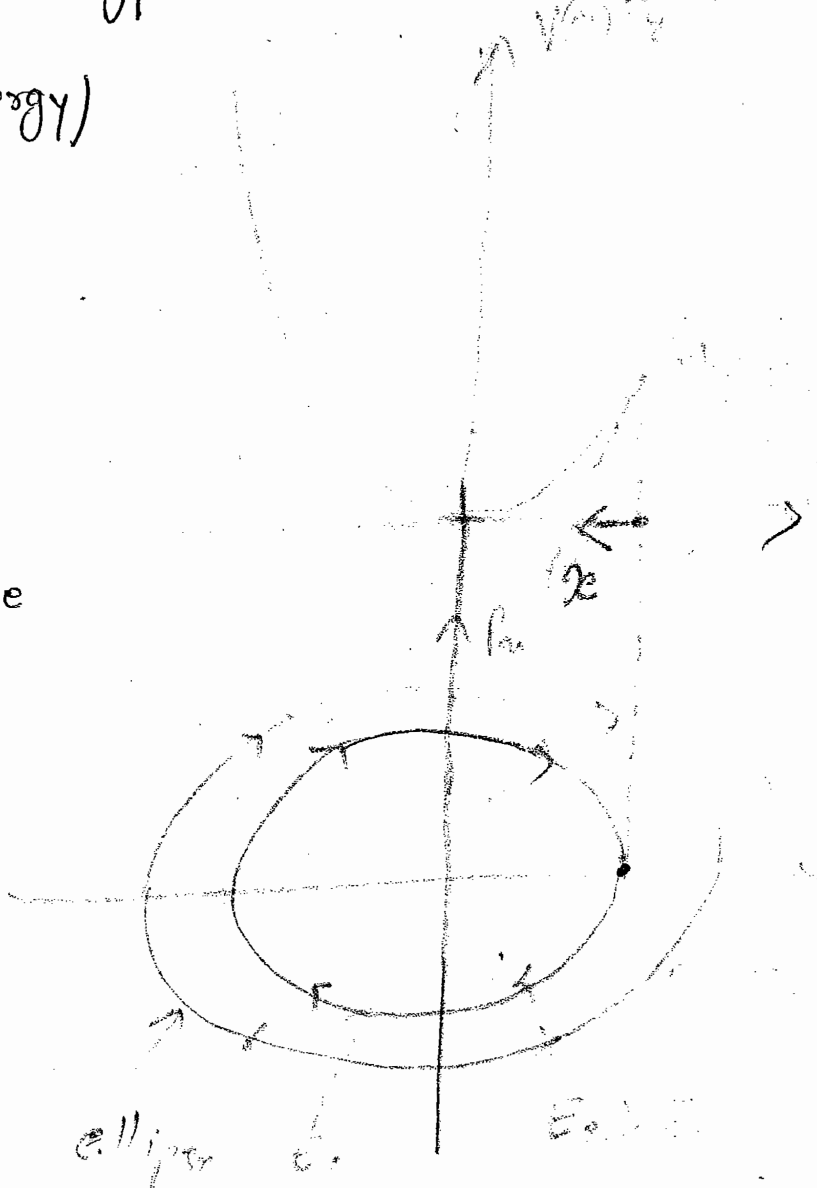
① Harmonic Oscillator:-

$$H = \frac{p_x^2}{2m} + \frac{1}{2}kx^2 = E \text{ (total Energy)}$$

If potential does not depends on velocity then H represents total energy.

$$H = \frac{p_x^2}{2m} + \frac{1}{2}kx^2 = E \text{ (energy)}$$

$$\frac{p_x^2}{2mE} + \frac{x^2}{\frac{2}{k}E} = 1$$



Here E does not change after $p_x \to -p_x$, $x \to -x$

∴ Symmetric about x and p_x both.

Here infinite potential well then phase space lines are closed curve.

* If potential well is infinite then phase space lines <u>inside</u> the well are closed lines if

energy less than height of well. phase space lines are closed lines.

Q. Particle moving under potential $V(x) = ax - bx^2$

Sol^n $H = \dfrac{p_x^2}{2m} + ax - bx^2 = E = \text{Constant}$

To draw $V(x)$ find zeros

$$V(x) = 0$$

$$ax - bx^2 = 0$$

$$x = 0, \quad x = \frac{a}{b}$$

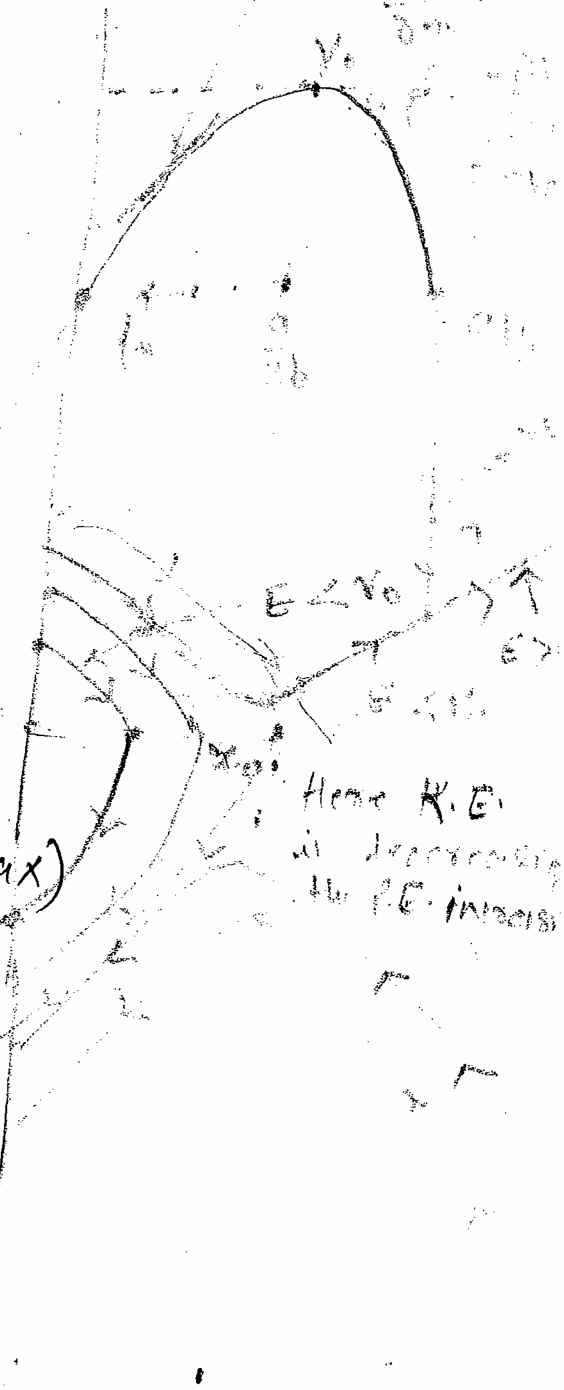
find extrimum points

$$\frac{dV}{dx} = 0$$

$$a - 2bx = 0$$

$$x = \frac{a}{2b}$$

$$\frac{d^2V}{dx^2} = -2b < 0 \quad (\text{max})$$



Q. Draw phase space trajectory for a particle moving under potential.

$$V(x) = ax - bx^3.$$

Sol^n To draw $V(x)$:-

find zeros :-

$$V(x) = 0$$

$$ax - bx^3 = 0$$

$$x(a - bx^2) = 0$$

$$x = 0 \quad \text{or} \quad a - bx^2 = 0$$

$$x = \pm\sqrt{\frac{a}{b}}$$

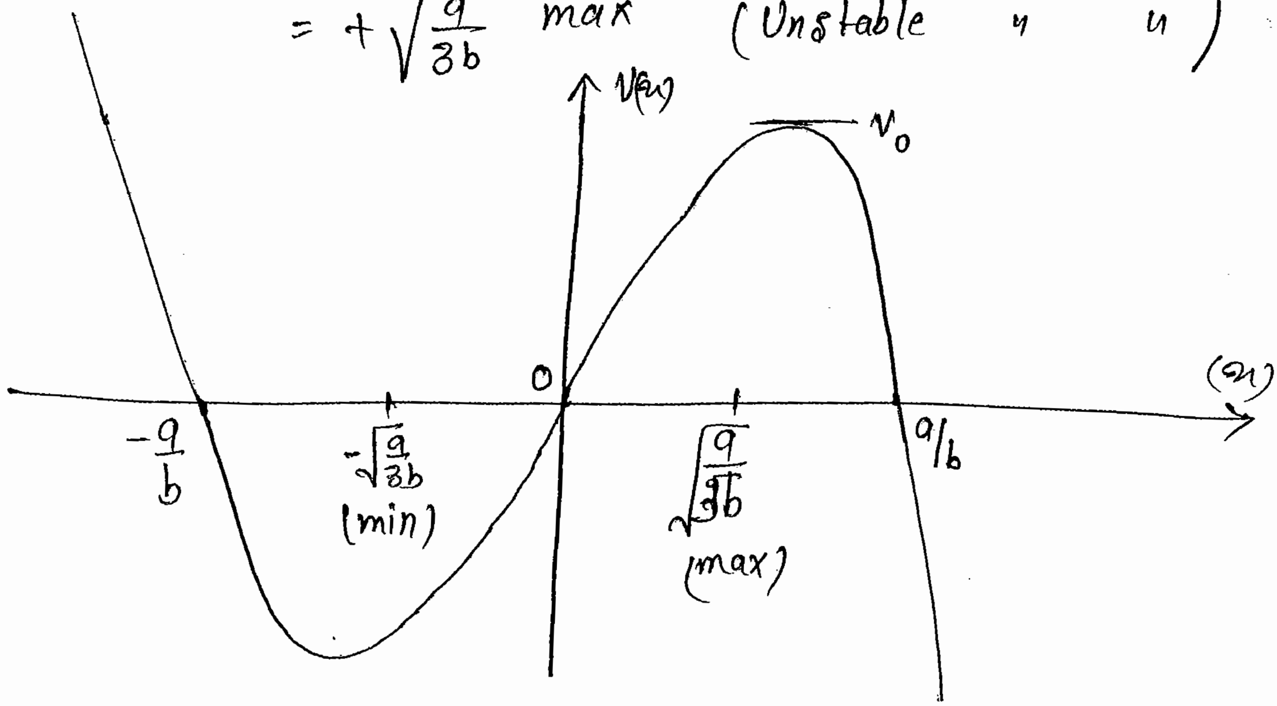
Extrimum points :-

$$\pm\sqrt{\frac{a}{3b}}$$

$$\frac{d^2V}{dx^2} = -6bx$$

$x = -\sqrt{\frac{a}{3b}}$ min (Stable equilibrium point)

$= +\sqrt{\frac{a}{3b}}$ max (Unstable " ")



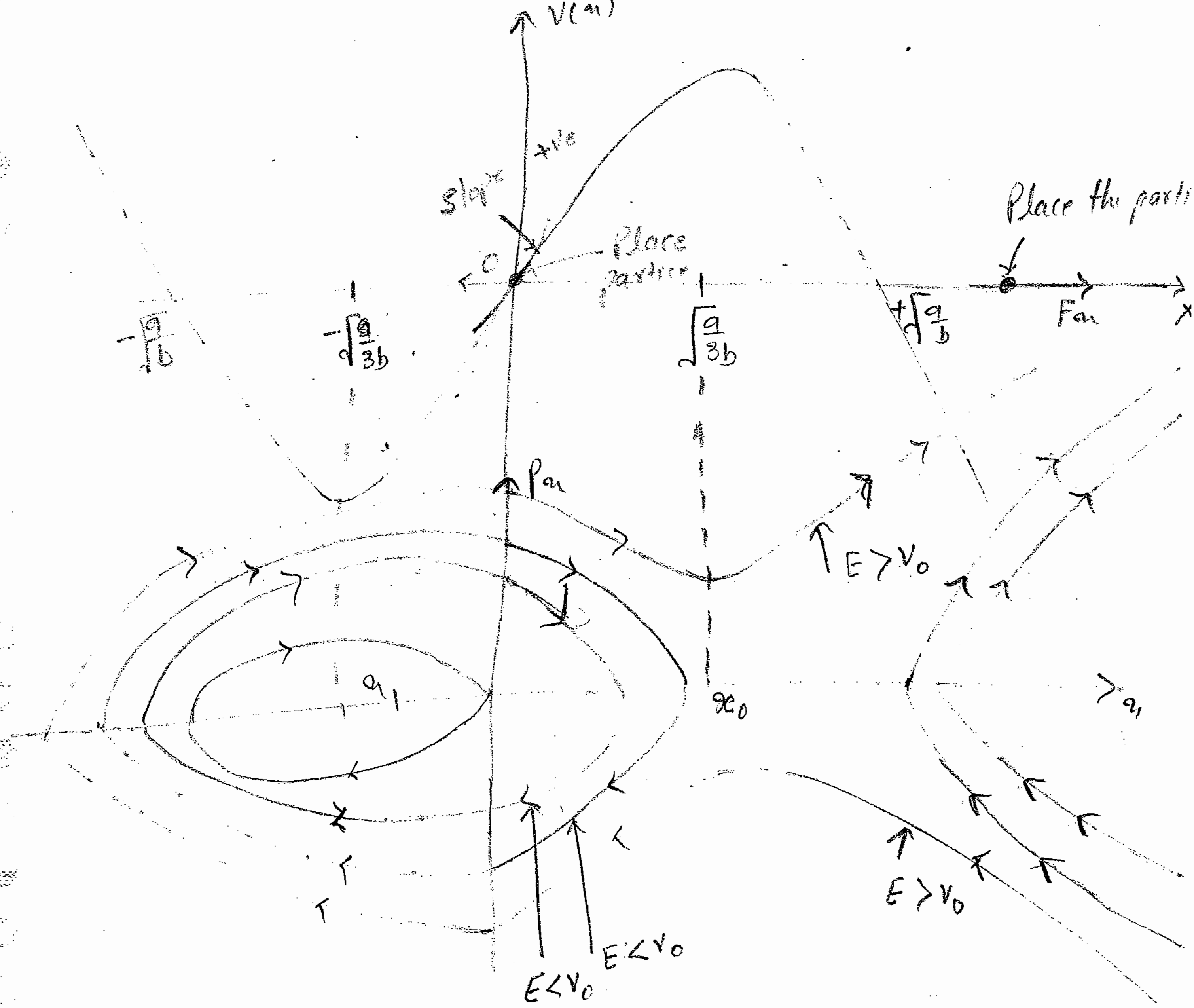
$\because \quad H = \frac{p_x^2}{2m} + ax - bx^3 = E$

Here $p_x \to -p_x$

No change so symmetric about x axis

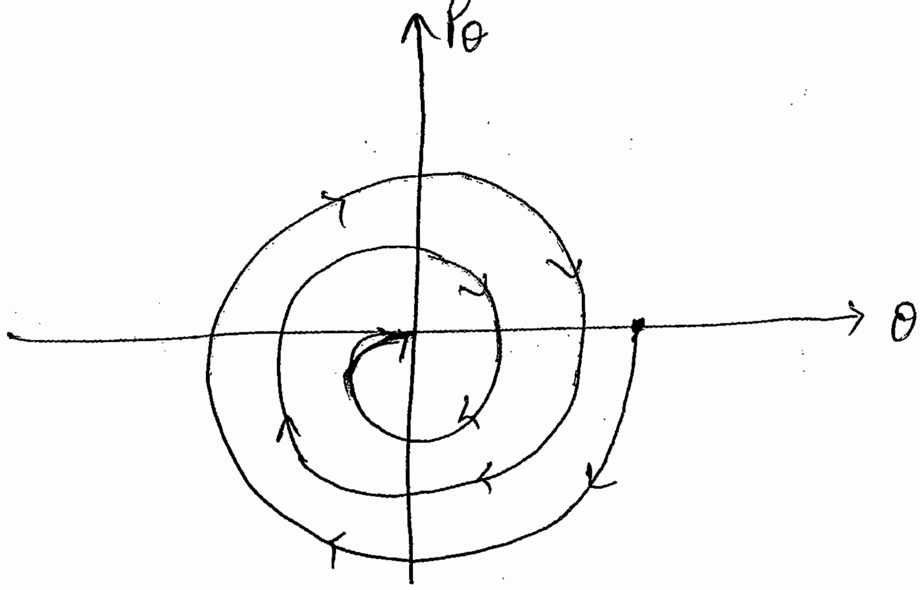
$x \to -x$

Change so does not symmetric about p_x-axis.

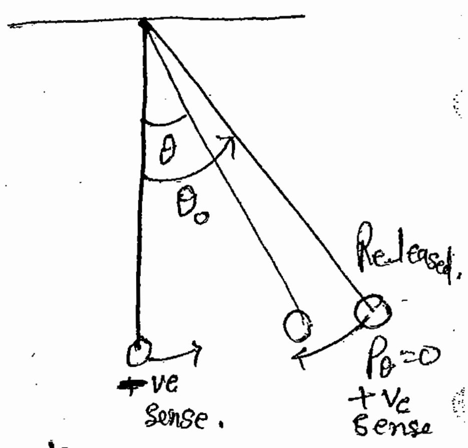


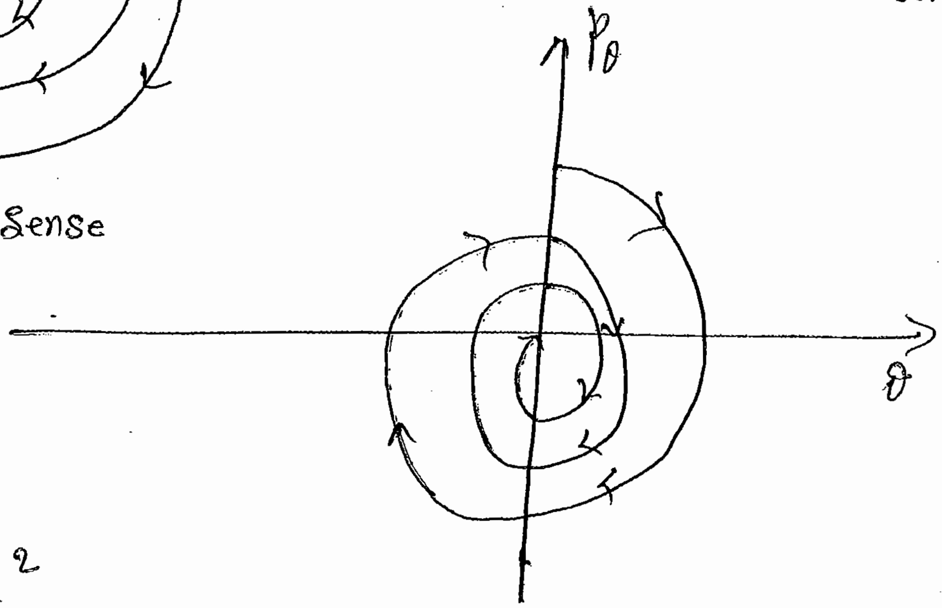
If a particle place a particle outside the potential well.

* Damped Oscillator (Pendulum) :-



for +ve sense

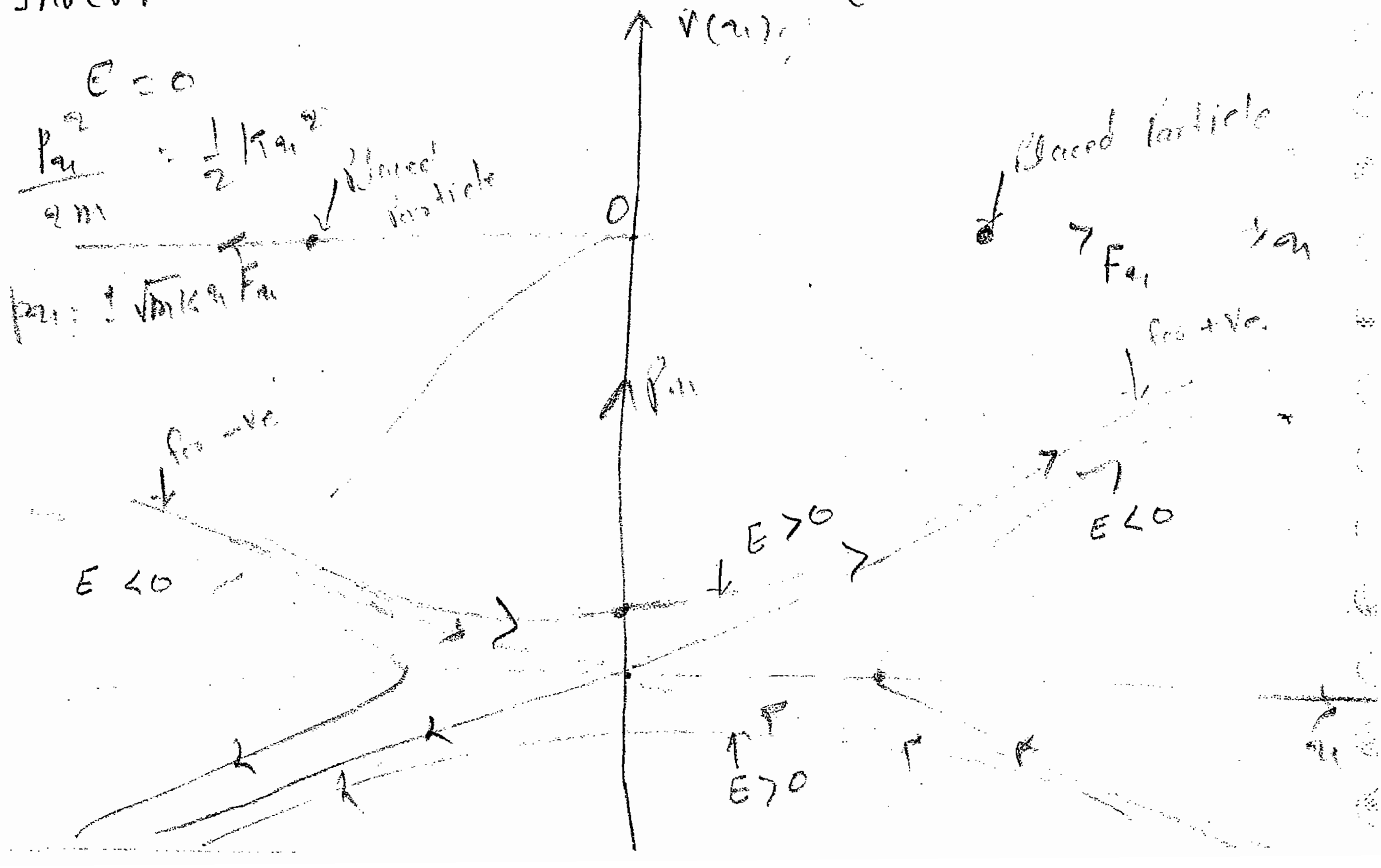




for -ve sense

$$H = \frac{p_x^2}{2m} - \frac{1}{2} k x^2$$

Inverted harmonic oscillator (No oscillation).



* Transformation :-

Point Transformation :- It is done in co-ordinate space.

$$(x,y,z) \longrightarrow (r,\theta,\phi)$$

It changes the form of Lagrangian but leave the form of Lagrangian's equation unchanged.

$$x, y, z \longrightarrow r, \theta, \phi$$

$$L \longrightarrow L'$$

$$\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \text{P.E. term} \longrightarrow \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2) - \text{P.E. term}.$$

Eq^n of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \longrightarrow \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{r}}\right) - \frac{\partial L'}{\partial r} = 0$$

* Gauge Transformation of Lagrangian :-

If Lagrangian is changed in following manner

$$L \longrightarrow L' = L \pm \frac{dF}{dt}$$

, where F is function of q_i and t only, Then, Lagrangian equation of motion does not change.

Example :-

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

Thus Lagrangian equation of motion.

$$m\ddot{x} + kx = 0$$

Let $L' = L + \dfrac{dF}{dt}$ and let $F = \beta x^2$

So $L' = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2 + 2\beta x \dot{x}$

Equation of motion.

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}(m\dot{x} + 2\beta x) + Kx - 2\beta \dot{x} = 0$$

$$m\ddot{x} + \cancel{2\beta \dot{x}} + Kx - \cancel{2\beta \dot{x}} = 0$$

$$\boxed{m\ddot{x} + Kx = 0}$$

In Guage transformation if we add any term which is total derivative of position and time, the equation of motion remains unchanged.

Note :-

Guage transformation of electromagnetic potentials does not change equation of motion because Lagrangian changes by total time derivative of function of $(\vec{r}, t)$.

Consider a particle (non-relativistic) moving in e.m. field (A, ϕ), So Lagrangian is -

$$\boxed{L = \frac{1}{2} mv^2 - q\phi + q\vec{A}\cdot\vec{V}}$$

Guage transformation of e.m. potential.

$$\vec{A}' = A + \vec{\nabla}\lambda(\vec{r}, t)$$

$$\phi' = \phi - \frac{\partial \lambda}{\partial t}$$

$$L' = \frac{1}{2}mv^2 - q\phi' + q\vec{A}'\cdot\vec{v}$$

$$L' = \frac{1}{2}mv^2 - q\phi + q\frac{\partial\lambda}{\partial t} + q\vec{A}\cdot\vec{v} + q(\vec{\nabla}\lambda)\cdot\vec{v}$$

$$L' = L + q\left[\frac{\partial\lambda}{\partial t} + (\vec{\nabla}\lambda\cdot\vec{v})\right]$$

$\because$ $f(x, y, z, t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} + \frac{\partial f}{\partial t}$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot(\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}) + \frac{\partial f}{\partial t}$$

$$\boxed{\frac{df}{dt} = \vec{\nabla}f\cdot\vec{v} - \frac{\partial f}{\partial t}}$$ → Total time derivative of any function.

So $$L' = L + q\frac{d\lambda}{dt}$$

$$\boxed{L' = L + \frac{d}{dt}(q\lambda)}$$

So equation of motion will not change.

B.A.2
Q.36

Sol^n

$$L = \frac{1}{2}mv^2 + e\vec{A}\cdot\vec{v} - e\phi$$

$$= \frac{1}{2}mv_i^2 + eA_iv_i - e\phi$$

$$p_i = \frac{\partial L}{\partial v_i} = mv_i + eA_i$$

$$\boxed{\vec{p} = m\vec{v} + e\vec{A}}$$

So first option is wroung.

Option (4) is correct. by previous relation.

A-11
Q.10

Sol^n

$L_1 = f(x, \dot{x})$, $L_2 = f(x, \dot{x}) + A(x\dot{y} \div y\dot{x})$

$L_2 = L_1 + A(x\dot{y} \div y\dot{x})$

$L_3 = f(x, \dot{x}) + A(x\dot{x} + y\dot{y})$

$L_4 = f(x, \dot{x}) + A(x\dot{y} + y\dot{x})$

$L_3 = L_1 + A\frac{d}{dt}(x^2 + y^2)$

Here we can not write this term in the form of total time derivative so L_1 and L_2 can't have same eq^n of motion.

Here we can write total time derivative then L_3 and L_4 have same eq^n of motion.

* Canonical Transformation :- {CT} :-

It is a point transformation in phase space.

CT : $(q_i, p_i) \longrightarrow (Q_i, P_i)$

Old co-ordinate & momentum — New Coordinate & momentum.

$H \longrightarrow H'$

Old Hamiltonian — New Hamiltonian.

* In canonical transformation form of Hamiltonian's eqn does not change.

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \frac{-\partial H}{\partial q_i} \longrightarrow \dot{Q}_i = \frac{\partial H'}{\partial P_i}, \quad \dot{P}_i = \frac{-\partial H'}{\partial Q_i}$$

* If form of Hamilton's equation does not change then form of Lagranges eqn of motion will also not change and this happens if Gauge condition is satisfied.

$$\boxed{L' = L - \frac{dF}{dt}(q_i, Q_i, t)}$$

$\because L = p_i \dot{q}_i - H$

So $$\boxed{P_i \dot{Q}_i - H' = p_i \dot{q}_i - H - \frac{df}{dt}(q_i, Q_i, t)}$$

This is condition for Canonical Transformation

Use $\frac{d}{dt}F(q_i, Q_i, t) = \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial Q_i}\dot{Q}_i + \frac{\partial F}{\partial t}$

$$P_i\frac{dQ}{dt} - H' = p_i\frac{dq_i}{dt} - H - \frac{\partial F}{\partial q_i}\frac{dq_i}{dt} - \frac{\partial F}{\partial Q_i}\frac{dQ_i}{dt} - \frac{\partial F}{\partial t}$$

$$P_i dQ - H'dt = p_i dq_i - H dt - \frac{\partial F}{\partial q_i}dq_i - \frac{\partial F}{\partial Q_i}dQ - \frac{\partial F}{\partial t}dt$$

Equating the coefficients of dq_i, dQ_i, dt from both side we get.

$$\left[p_i - \frac{\partial F}{\partial q_i}\right]dq_i = 0 \Rightarrow p_i - \frac{\partial F}{\partial q_i} = 0$$

$$\Rightarrow \boxed{p_i = \frac{\partial F}{\partial q_i}}$$

and $$\left[P_i + \frac{\partial F}{\partial Q_i}\right]dQ_i = 0 \Rightarrow P_i + \frac{\partial F}{\partial Q_i} = 0$$

$$\Rightarrow \boxed{P_i = -\frac{\partial F}{\partial Q_i}}$$

and $$\boxed{H' = H + \frac{\partial F}{\partial t}} \quad \text{———} (2)$$

$$\left.\begin{aligned} P_i &= -\frac{\partial F}{\partial Q_i} \\ H' &= H + \frac{\partial F}{\partial t} \\ p_i &= \frac{\partial F}{\partial q_i}(q_i, Q_i, t) \end{aligned}\right\} \text{———} (1)$$

If $q_i, p_i \longrightarrow Q_i$ is C.T. then we can write a function $F(q_i, Q_i, t)$ such that such that eqⁿ

① will be satisfies. Here $F(q_i, Q_i, t)$ is a generating function.

Now we can use eq^n ① to show that in C.T. Poisson bracket

$$\boxed{\{Q_i, P_i\}_{q_i p_i} = 1 \quad \text{and} \quad \{q_i, p_i\}_{Q_i, P_i} = 1}$$

← for checking function is C.T. or no, we satisfies this poission bracket if ① bracket is satisfies then second also satisfies so we can check easily

* Jacobian of Transformation :-

Jacobian is defined for all type of function.

Here, in this case - Jacobian of Transformation is one. but it is valid only for Canonical form.

$$\boxed{dp_i dq_i = dP_i dQ_i}$$

$$dp_1 dp_2 \ldots dq_1 dq_2 \ldots = dP_1 dP_2 \ldots dQ_1 dQ_2 \ldots$$

For Ex -
$$x, y, z \longrightarrow r, \theta, \phi$$
$$dx, dy dz \longrightarrow dr\, d\theta\, d\phi$$
$$dx\, dy\, dz \longrightarrow J\, dr\, d\theta\, d\phi$$

When $J = r^2 \sin\theta$ is Jacobian of transformat

Q. Consider a transformation $x, P_x \longrightarrow x', P_x'$ such that

$$x' = x\cos\theta - P_x \sin\theta$$
$$P_x' = x\sin\theta + P_x \cos\theta$$

is this transformation a C.T ?

P.B. $\{x', p_x'\}_{x, p_x} = 1$

$$\Rightarrow (\cos\theta - 0)(0 + \cos\theta) - (-\sin\theta)\sin\theta = 1$$

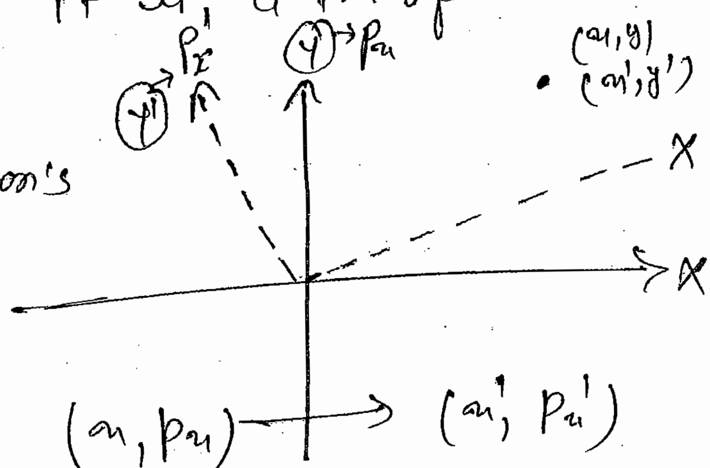
$$\cos^2\theta + \sin^2\theta = 1$$

$$\boxed{1 = 1}$$

So given transformation is canonical transformation

Meaning of C.T. is it is a transformation in phase space.

That need the hamilton's eqn of motion remain's unchange.



$(x, p_x) \longrightarrow (x', p_x')$

$$x' = x\cos\theta - p_x\sin\theta$$

$$p_x' = x\sin\theta + p_x\cos\theta$$

Note :- Why we do this transformation.

$\cancel{H = \frac{p^2}{2m} + \frac{1}{2}kx^2 - bx}$

Say $V(x) = \frac{1}{2}kx^2 - bx$

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 - bx$$

$$H = \frac{p^2}{2m} + \frac{1}{2}k\left[\left(x^2 - \frac{2b}{k}x\right) + \left(\frac{b}{k}\right)^2 - \left(\frac{b}{k}\right)^2\right]$$

$$\Rightarrow H = \frac{p^2}{2m} + \frac{1}{2}k\left(x - \frac{b}{k}\right)^2 - \frac{b^2}{2k}$$

$$\left.\begin{array}{l} x' = \left(x - \frac{b}{K}\right) \\ P_x' = p_x \end{array}\right\}$$ { C.T. is a general transformation }

So $$\boxed{E = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{b^2}{2K}}$$

We can also solve this ~~transfor~~ problem by perturbati theory but by simplisity we can do this transforma to solve prob. easily. So ~~th~~ this is a canonical transformation in phase space.

Q. If transformation

$$P = aq + bp$$
$$Q = cq + dp$$

is a C.T. then what is relation b/w constants a, b, c & d ?

Solⁿ

$$\{Q, P\}_{q,p} = 1$$

$$cb - da = 1 \Rightarrow \boxed{bc - ad = 1}$$ Ans

Q. If transformation $Q = q^\alpha \sin\beta p$ and $P = q^\alpha \cos\beta p$ is a C.T. What is the relation of α and β.?

Solⁿ

$$\{Q, P\}_{q,p} = 1$$

$$\{q^\alpha \sin\beta p,\ q^\alpha \cos\beta p\}_{q,p} = 1$$

$$-\alpha q^{\alpha-1}\sin\beta p \cdot \beta q^\alpha \sin\beta p - \beta q^\alpha \cos\beta p \cdot \alpha q^{\alpha-1}\cos\beta p = 1$$

$$-\alpha\beta q^{2\alpha-1} = 1 \cdot q^0 \quad \text{—} (1)$$

"If we change q then Right side is not change so left side should also

equating the power

$$2\alpha - 1 = 0$$

$$\boxed{\alpha = \frac{1}{2}}$$

from (1)

$$-\alpha \beta q = 1$$

$$-\frac{1}{2}\beta = 1$$

$$\boxed{\beta = -2}$$

* Generating function ⇒

It is a function of one new variable and one old variable which puts a condition on transformation in phase space to make it canonical.

If $F(q_i, Q_i)$ then it is related to co-ordinates as -

$$p_i = \frac{\partial F}{\partial q_i}(q_i, Q_i)$$

$$P_i = -\frac{\partial F(q_i, Q_i)}{\partial Q_i}$$

q, Q
p, Q
p, P
q, P

* Legendre Transformation (Mathematics) :-

This is a most general transformation.

It changes one variable into other variable due to which form of the function changes.

If there is a function which depends on the n.

$$x \longrightarrow s$$

$$F(x) \longrightarrow G(s)$$

$$\boxed{G(s) = F(x) \pm xs}$$ ← This is Legendre transformation.

Here in mathematics + or −ve sign are taken by maxima or minima. we want $G(s)$ is maximum. So when $F(x)$ is −ve so xs is also −ve when $F(x)$ is +ve then xs is taken as positive.
But in Physics the consideration of +ve or −ve sign's concepts are different from mathematics. which is specified below.

* Generating functions :-

$$q_i \longrightarrow p_i$$

$$F_1(q_i, Q_i) \longrightarrow F_2(p_i, Q_i) \quad \{\text{Legendre Trans.}\}$$

$$F_2(p_i, Q_i) \longrightarrow F_1(q_i, Q_i) \pm q_i p_i$$

Diff. w.r. to q_i which is not present in F_2

$$0 = \frac{\partial F_1}{\partial q_i} \pm p_i$$

we will get previous relation $\left(p_i = \frac{\partial F}{\partial q_i}\right)$ if we use $(-)$ve sign.

i.e. ⇒ $$\boxed{F_2(p_i, Q_i) \longrightarrow F_1(q_i, Q_i) - q_i p_i}$$

Similarly, $\boxed{F_2(p_i, Q_i) = \underline{F_1(q_i Q_i)} - q_i p_i}$

If we put $F_1(q, Q) = F_2(P_i, Q_i) + q_i p_i$ in the relation.

$$P_i \dot{Q}_i - H' = p_i \dot{q}_i - H + \frac{dF_1(q, Q)}{dt}$$

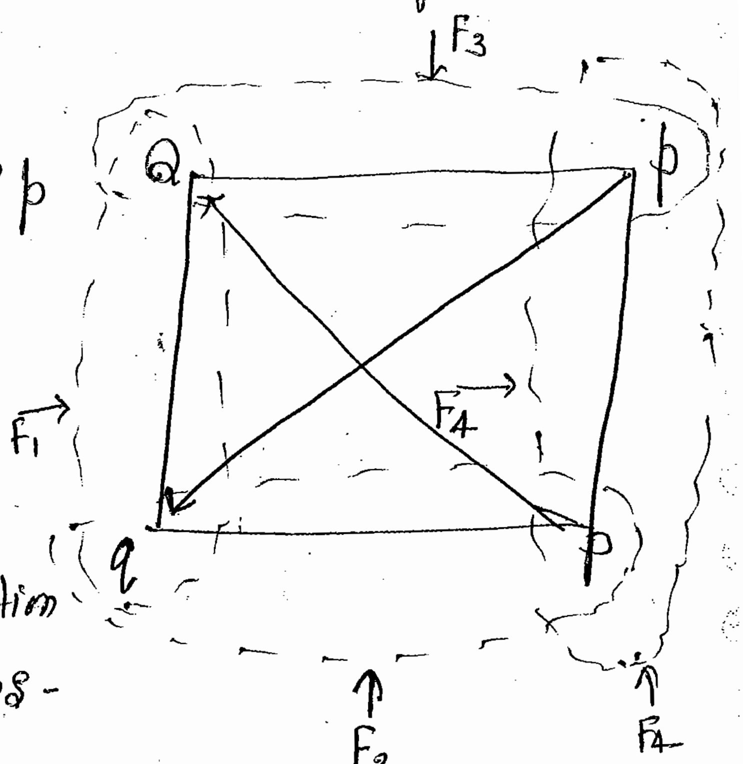
then we get relation for F_2.

Short Trick :- (No physical background)

It is a method to guess relation for different Generating function.

Steps:-

(I) Draw a rectangle $Q\,q\,P\,p$



(II) Choose any two variable side wise.

(III) Relation for Generating function can be obtained as follows -

(a) Draw an arrow from the variable which is to be obtained towards diagonally opposite variable

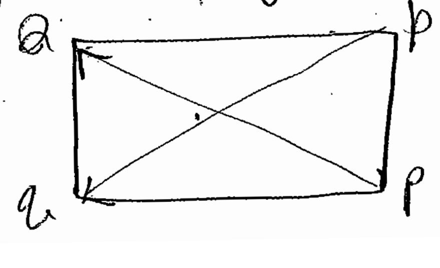
If arrow points up use negative sign

If arrow points down use positive sign.

Ex-

$$F_1 = F_1(q, Q)$$

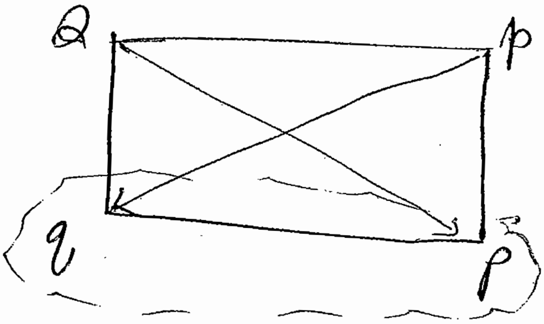
$$p_i = \frac{\partial F_1}{\partial q_i}, \quad P = -\frac{\partial F_1}{\partial Q_i}$$



* Relation for $F_2(q, P)$:-

$$p = \frac{\partial F_2}{\partial q}$$

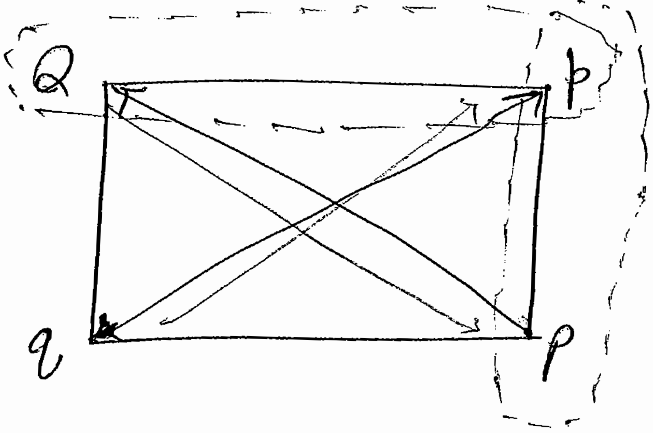
$$Q = \frac{\partial F_2}{\partial P}$$



(*) Relation for $F_3(p, Q)$:-

$$q = -\frac{\partial F_3}{\partial p}$$

$$P = -\frac{\partial F_3}{\partial Q}$$



$$Q = = \frac{\partial F_4}{\partial P}$$

$$q = -\frac{\partial F_4}{\partial p}$$

---

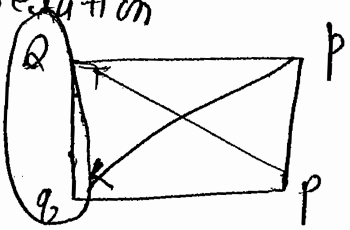
*Imp How to obtain generating function from a given transformation :-

(1) If we have to obtain $F(q, Q)$

(2) Then use its relation

$$p = \frac{\partial F}{\partial q}$$

$$P = -\frac{\partial F}{\partial Q}$$



(3) Express L.H.S. of above relation in trems of chosen variables by using given transformation relation.

(4) Integrate the relation for F.

(5) Combine the two value of F obtain (write common ter once only).

BA-3

Q.96

Sol^n

$$(q, p) \longrightarrow (Q, P)$$

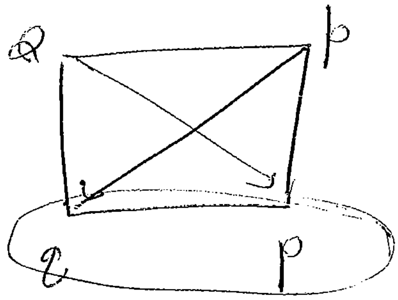
$$Q = q^2$$

$$P = \frac{p}{2q}$$

We have to find $F(q, P)$

$$Q = \frac{\partial F}{\partial P} \quad \text{—(i)}$$

$$p = \frac{\partial F}{\partial q} \quad \text{—(ii)}$$



from (i)

$$q^2 = \frac{\partial F}{\partial P}$$

$$F = q^2 P + G_1(q) \quad \text{—(iii)}$$

from (2)

$$p = \frac{\partial F}{\partial q}$$

$$2Pq = \frac{\partial F}{\partial q}$$

Integrate :-

$$F = q^2 P + G_2(P) \quad \text{—(iv)}$$

Using (iii) and (iv)

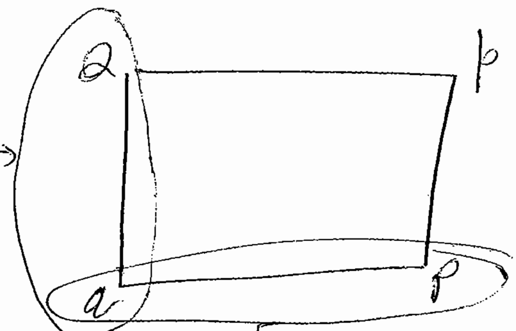
$$\boxed{F = q^2 P}$$

Ans

Q. 2

Soln Identity Transformation -

$Q = q$
$P = p$



$f_1(q, Q)$

$p = \frac{\partial F(q,Q)}{\partial q}$

$P = -\frac{\partial F}{\partial Q}$

Here we can not express in terms of choosen variable.
So we can not solve it

$p = \frac{\partial F_2(q, P)}{\partial q}$

$Q = \frac{\partial F_2}{\partial P}$

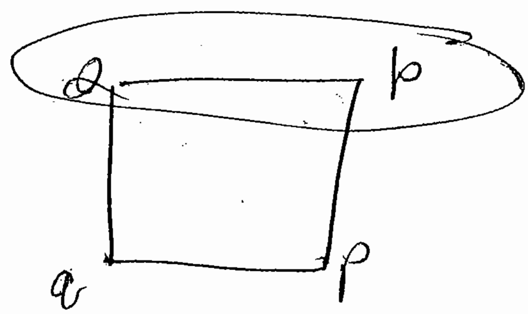
Q. Which of the following G.F. gives identity transformation

$F_1 = qP$, $F_2 = -Qp$

(a) Only F_1 (b) only F_2 (c) both (d) non

Soln $P = \frac{-\partial F_2}{\partial Q_1} = p$

$q = -\frac{\partial F_2}{\partial p} = Q$



Ques 6

Sol^n $Q_i = p_i \tan t$

$P_i = q_i \tan t$

We have to find $F(q_i, Q_i)$

$$p_i = \frac{\partial F}{\partial q_i} \longrightarrow \frac{Q_i}{\tan t} = \frac{\partial F}{\partial q_i} \Rightarrow f = \frac{Q_i q_i}{\tan t} + C(Q)$$

$$P_i = \frac{-\partial F}{\partial Q_i} \longrightarrow \frac{\partial F}{\partial Q_i} = -q \tan t \Rightarrow f = -Q q_i \tan t + C(q_i)$$

$$\therefore F = \frac{Q_i q_i}{\tan t} + Q_i q_i \tan t$$

$$= Q_i q_i \left[\frac{\cos t}{\sin t} - \frac{\sin t}{\cos t}\right]$$

$$= 2 Q_i q_i \frac{[\cos 2t]}{2 \sin t \cos t}$$

* Rotation in phase space :-

$$(q, p) \longrightarrow (Q, P)$$

$$\begin{pmatrix} Q \\ P \end{pmatrix} = R \begin{pmatrix} q \\ p \end{pmatrix}$$

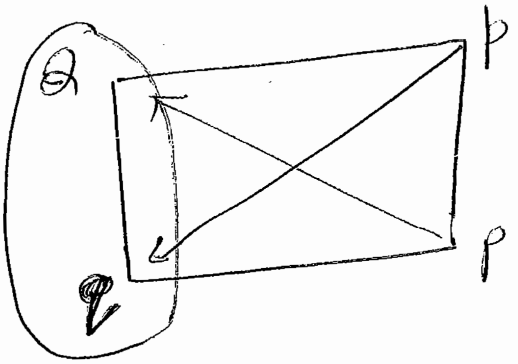
$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

$$\left.\begin{aligned} Q &= q\cos\theta - p\sin\theta \\ P &= q\sin\theta + p\cos\theta \end{aligned}\right\}$$ Transformation Relation

We have to find $F(q,Q)$:-

$$p = \frac{\partial F}{\partial q}$$

$$P = -\frac{\partial F}{\partial Q}$$



from ① relation -

$$\frac{q\cos\theta - Q}{\sin\theta} = \frac{\partial F}{\partial q}$$

$$F = \frac{q^2}{2}\cot\theta - qQ\,\mathrm{cosec}\,\theta + c_1(Q)$$

from ② relation -

$$P = \frac{-\partial F}{\partial Q}$$

$$-q\sin\theta - \cos\theta\left[\frac{q\cos\theta - \theta}{\sin\theta}\right] = \frac{-\partial F}{\partial Q}$$

$$\Rightarrow c_2(q) - qQ\sin\theta - \cot\theta\left[qQ\cos\theta - \frac{Q^2}{2}\right] = 0$$

$$\Rightarrow -qQ\sin\theta + \cot\theta\left[qQ\cot\theta - \frac{Q^2}{2}\right] + c_2(q).$$

$$\Rightarrow -qQ\left[\sin\theta + \frac{\cos^2\theta}{\sin\theta}\right] + \frac{Q^2}{2}\cot\theta + c_2(q).$$

$$\boxed{F = -qQ\,\mathrm{cosec}\,\theta + \frac{1}{2}(q^2+Q^2)\cot\theta}$$

Ans.

H-12
Q.5

Soln Given $H = \frac{p^2}{2m} + mgq$

$$F(q,Q,t) = \frac{1}{3m^2 g}\left[2m^2 g(Q-q)\right]^{3/2} = \frac{1}{3m^2 g}(2m^2 g)^{3/2} \times (Q-q)^{3/2}$$

Here F is not depends on time explicitely.

$$H'(P,Q) = H(p,q) + \underbrace{\left(\frac{\partial F}{\partial t}\right)}_{0}$$ ['F' is not explicit function of t]

So $\boxed{H' = H}$

$$H = (P,Q) = \frac{p^2}{2m} + mgq$$

Express p & q in new variable

Relation of G.F. :-

$$p = \frac{\partial F}{\partial q} = \frac{-1}{3m^2 g}(2m^2 g)^{3/2}\frac{3}{2}(Q-q)^{1/2}(-1)$$

$$P = \frac{-\partial F}{\partial q} = +\frac{1}{3m^2 g}(2m^2 g)^{3/2}\frac{3}{2}(Q-q)(1)$$

So $\boxed{p = P}$

$$P = \frac{1}{\cancel{3}m^2 g}(2m^2 g)^{3/2}\frac{\cancel{3}}{2}(Q-q)^{1/2}$$

Squairing on both side :-

$$P^2 = \frac{1}{4a^2 g^2} \cdot 8m^6 g^3 (Q-q).$$

$$\boxed{q = Q - \frac{p^2}{2m^2 g}}$$

So New hamiltonian is -

$$H'(PQ) = \frac{p^2}{2m} + mgq$$

$$\boxed{H'(PQ) = \frac{p^2}{2m} + mg\left(Q - \frac{p^2}{2m^2 g}\right)}$$

A-12

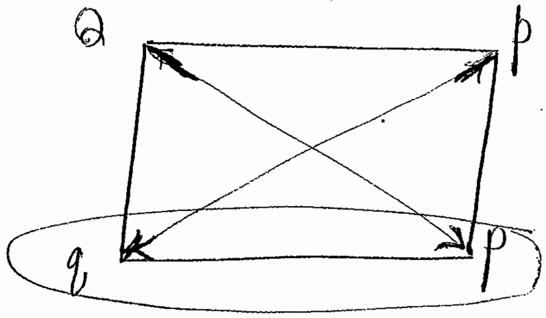
Q.11 Find generating function $F(P, q)$ for transformation $p = \frac{1}{Q}$, $q = PQ^2$ is - ?

(a) $\sqrt{Pq}$ (b) $-\sqrt{Pq}$ (c) $2\sqrt{Pq}$ (d) $-2\sqrt{Pq}$.

Sol^n Given transformation relation is -

$$p = \frac{1}{Q}$$

$$q = PQ^2$$



$$p = \frac{\partial F(q, P)}{\partial q} \quad \text{--- (I)}$$

$$Q = \frac{\partial F(q, P)}{\partial P} \quad \text{--- (II)}$$

$$Q = \sqrt{\frac{q}{P}}$$

$$p = \frac{1}{Q}$$

$$p = \sqrt{\frac{P}{q}}$$

from (I) relation -

$$\sqrt{\frac{P}{q}} = \frac{\partial F}{\partial q}$$

$$\sqrt{P}\, 2\sqrt{q} + c_1(P) = F \quad \text{--- (a)}$$

from (II)

$$\sqrt{\frac{q}{P}} = \frac{\partial F}{\partial P} \Rightarrow F = \sqrt{q} \cdot 2\sqrt{P} + c_2(q) \quad \text{--- (b)}$$

So from (a) & (b) $\boxed{F = 2\sqrt{Pq}}$ Ans

Q9 Values of 'a' and 'b' for which following transformation $Q = (2q)^a \cos bp$, $P = (2q)^a \sin p$ is canonical are -

(a) $a = \frac{1}{2}, b = 1$ (b) $a = 2, b = \frac{1}{2}$ (c) $a = 1, b = 1$, (d) $a = \frac{1}{2}, b = \frac{1}{2}$

Sol^n For canonical possion bracket should be one.

So $\{Q, P\}_{q,p} = 1$

$$\Rightarrow \{(2q)^a \cos^b p, (2q)^a \sin^b p\}_{q,p} = 1$$

$$\Rightarrow 2a(2q)^{a-1} \cos^b p \cdot (2q)^a b \sin^{b-1} p \cos p + (2q)^a b \cos^{b-1} p \sin p \times 2a(2q)^a \sin^b p = 1$$

$$\Rightarrow 2ab(2q)^{2a-1} \cos^{b-1} p \sin^{b-1} p [\cos^2 p + \sin^2 p] = 1$$

$$2ab(2q)^{2a-1} (\cos p \sin p)^{b-1} = 1$$

$\because$ R.H.S is co-ordinate independent $\therefore$ L.H.S. should also be co-ordinate independent.

So $2a - 1 = 0$ and $b - 1 = 0$

$\boxed{a = \frac{1}{2}}$ $\boxed{b = 1}$

* Hamilton Jacobi Theory:-

In this theory a canonical transformation is done in such a way that new hamiltonian becomes zero. And G.F. for this C.T. is chosen to be $F[q, P]$

$$q, p \xrightarrow{C.T.} Q, P$$

$$H \xrightarrow[F(q,P)]{} H' = 0$$

Hamiltons equation in new co-ordinate-

$$\dot{Q} = \frac{\partial H'}{\partial P} = 0 \qquad \{\because H' = 0$$

$$\dot{P} = -\frac{\partial H'}{\partial Q} = 0$$

$\Rightarrow$ Q = constant
& P = constant

Importance of $F(q, P, t)$ in H.J. theory.

Total time derivative of F.

$$\frac{dF}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial P}\underbrace{\dot{P}}_{=0} + \frac{\partial F}{\partial t}$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial t}$$

relation for $f(q, P, t)$

$$p = \frac{\partial F}{\partial q}$$

$$Q = \frac{\partial F}{\partial P}$$

$$H' = H + \frac{\partial F}{\partial t}$$
$$\overset{\shortparallel}{0}$$

$$\frac{\partial F}{\partial t} = -H$$

$$\frac{dF}{dt} = p\dot{q} - H$$

$$\boxed{\frac{dF}{dt} = L}$$ — Lagrangian

$$\boxed{F = \int L\,dt} = S$$

Action (S)

In H-J theory F is equal to action (S).

Imp.
* H-J Equation :-

$$\boxed{H + \frac{\partial S}{\partial t} = 0}$$

In Hamilton every place of p we will write -

$$\boxed{p = \frac{\partial S}{\partial q}}$$

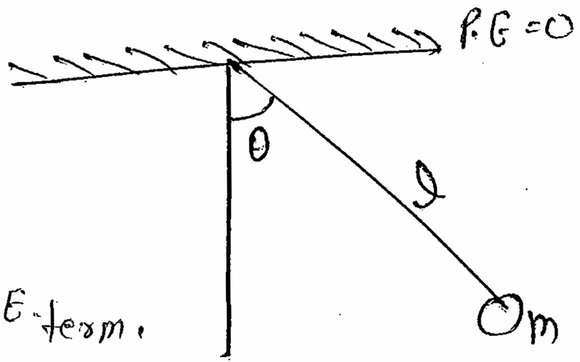
Q. Write hamilton-Jacobi eq^n for simple pendulum.

(r, θ)

$r = l$

$\dot{r} = 0$

$p_r = 0$



$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \text{P.E. term.}$$
$$= \frac{p_\theta^2}{2ml^2} - mgl\cos\theta$$

$$p_\theta = \left(\frac{\partial S}{\partial \theta}\right)$$

So H.J. eq^n for simple pendulum

$$H + \frac{\partial S}{\partial t} = 0$$

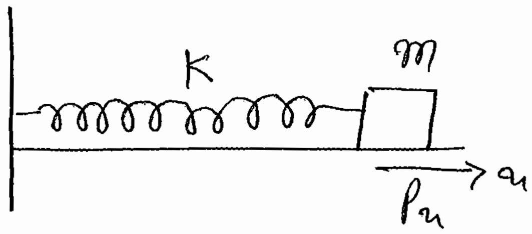
$$\boxed{\frac{1}{2ml^2}\left(\frac{\partial S}{\partial \theta}\right)^2 - mgl\cos\theta + \frac{\partial S}{\partial t} = 0} \quad *$$

$\Downarrow$

Non-linear P.D.E. (Solution of Non-linear partial diff. eq^n can be found by variable separation on sumation not multiplication)

* H-J Equation for S.H.O. :-

$$H + \frac{\partial S}{\partial t} = 0 \quad , \quad p = \frac{\partial S}{\partial q}$$



$$H = \frac{p_x^2}{2m} + \frac{1}{2}Kx^2$$

$$= \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}Kx^2$$

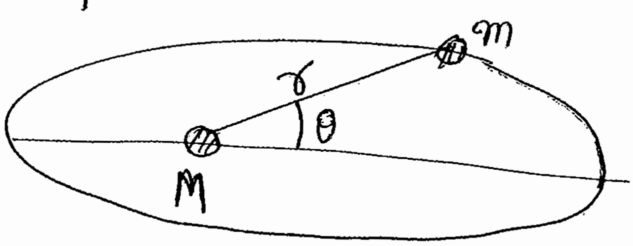
Here H-J eq^n -

$$\boxed{\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}Kx^2 + \frac{\partial S}{\partial t} = 0}$$

Ques: H-J equation for planatary motion.

Sol^n:-

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{GMm}{r}$$



$$p_r = \left(\frac{\partial S}{\partial r}\right), \quad p_\theta = \frac{\partial S}{\partial \theta}$$

$$\boxed{\frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta}\right)^2 - \frac{GMm}{r} + \frac{\partial S}{\partial t} = 0}$$

Non-Linear P.D.E.

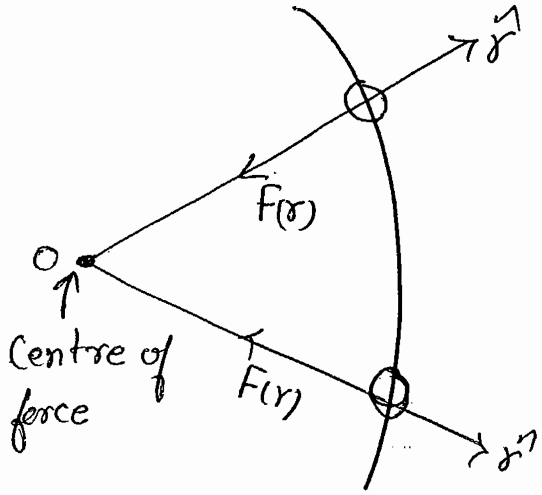
$$S = S(r, \theta, t)$$

$$S = S_1(r) + S_2(\theta), + S_3(t).$$

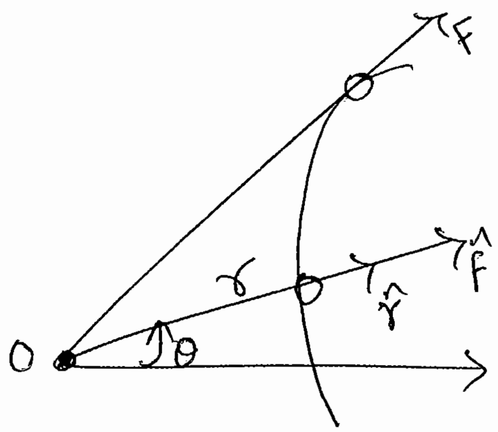
Central forces Motion :-

force is directed towards or away from a point is central force.

$$\boxed{f = \pm f(r)\,\hat{r}}$$



- When $\vec{F}$ and $\hat{r}$ is in same direction then we use +ve sign.
- When $\vec{F}$ and $\hat{r}$ is in opposite direction then we use −ve sign.



* Properties :-

$$\vec{\nabla}\times\vec{F} = 0$$

⇒ Central force is conservative.
⇒ Total energy is conserved.

$\boxed{\vec{\tau}_0 = 0}$, $\boxed{L_0 \text{ is conserved.}}$

When angular momentum is conserv then it must be two directional problem.

$\vec{F}$ is conservative. So we can define P.E. ($V(r)$)

$$\boxed{V(r) = -\int f(r)\,dr}$$

, $f(r)$ is +ve for repulsive force
, $f(r)$ is −ve for attractive force

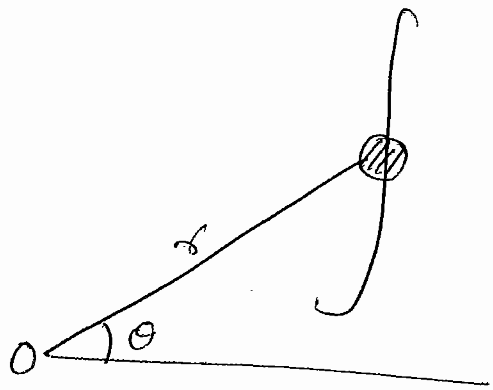
$$\vec{f} = -\vec{\nabla} V(r)$$

$$\boxed{f = -\frac{\partial}{\partial r} V(r)}$$

Equation of motion under central force :-

$$L = T - V$$

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$



Here L is independent of t so total energy is conserved and L is independent of θ so P_θ (angular momentum) is conserved.

Equation of motion :-

r- equation :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$m\ddot{r} - mr\dot{\theta} + \frac{\partial V}{\partial r} = 0$$

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 - f(r) = 0}$$ Imp. ← radial eqn of motion.

θ- equation :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}(mr^2\dot{\theta}) - 0 = 0$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0$$

$$\boxed{mr^2\dot{\theta} = \text{Constant}}$$ Imp

$$\boxed{2\dot{r}\dot{\theta} + r\ddot{\theta} = 0}$$ Imp.

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \boxed{mr^2\dot{\theta} = l} \text{ Angular momentum.}$$

$$\boxed{\dot{\theta} = \frac{l}{mr^2}}$$

Q. Transverse velocity of a particle under central force varies with r as -

(a) $\frac{1}{r}$ (b) $\frac{1}{r^2}$ (c) r (d) $\frac{1}{r^2}$

Sol^n Transverse velocity $= r\dot{\theta}$ $\left\{\because \dot{\theta} = \frac{l}{mr^2}\right\}$

$= r \cdot \frac{l}{mr^2}$ $\left\{ K.E. = \frac{1}{2} m(\underbrace{\dot{r}^2}_{\text{Radial Velocity}} + \underbrace{r^2\dot{\theta}^2}_{\text{Trans veloc}}) \right.$

$= \frac{l}{m} \cdot \frac{1}{r}$

Hence transverse velocity $\propto \frac{1}{r}$

* Equation of path in central force motion :-

↓

is independent of ~~path~~ time.

remove t from equation of motion.

Introduce $u = \frac{1}{r}$

This leads to following eq^n

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{mf}{l^2u^2}}$$

⇓

Diff. eq^n of path

$u = \frac{1}{r} \Rightarrow r = \frac{1}{u}$

$\dot{r} = -\frac{1}{u^2}\frac{du}{dt}$

$= -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$

$= -r^2\frac{du}{d\theta}\cdot\dot{\theta}$

$= -r^2\frac{du}{d\theta}\cdot\frac{l}{mr^2}$

$= -l\frac{du}{d\theta}$

Use :- It is use to find $f(r)$ when relation b/w r and θ is given.

Q. If $r = a\cos\theta$ for a particle moving under central force what is force law?

Sol^n

$$u = \frac{1}{r}$$

$$u = \frac{1}{a}\sec\theta$$

$$\frac{du}{d\theta} = \frac{1}{a}\sec\theta\tan\theta$$

$$\frac{d^2u}{d\theta^2} = \frac{1}{a}\left[\sec\theta\tan^2\theta + \sec^3\theta\right]$$

Differential equation of path -

$$\frac{d^2u}{d\theta^2} + u = \frac{-mf}{l^2u^2}$$

$$u\tan^2\theta + u\sec^2\theta + u = \frac{-mf}{l^2u^2}$$

$$2u\sec^2\theta = \frac{-mf}{l^2u^2}$$

$$f = -\frac{2l^2u^3\sec^2\theta}{m}$$

$$f = -\frac{2l^2a^2}{m}u^5$$

$$f = \frac{-2l^2a^2}{m}\frac{1}{r^5}$$

$$\boxed{f \propto \frac{1}{r^5}} \leftarrow \text{Force Law}$$

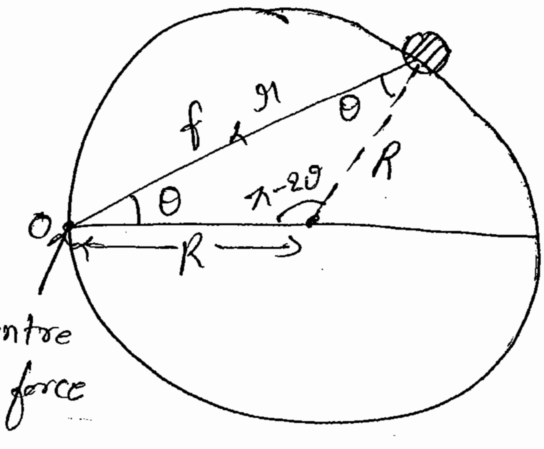
Note (i) If $r = a\sin\theta$

$$\Rightarrow \boxed{f \propto \frac{1}{r^5}}$$

(ii) if $r^n = a\cos n\theta$ or $r^n = a\sin n\theta$ $\boxed{f \propto \frac{1}{r^{2n+3}}}$

Imp.
Q. A particle is moving in circle of radius R under a force which is always directed towards a point on periferi on circle
(i) What is the force law. (ii) What is total energy.

Sol^n



By sine law

$$\frac{r}{\sin(\pi-2\theta)} = \frac{R}{\sin\theta}$$

$$\frac{r}{\sin 2\theta} = \frac{R}{\sin\theta} \Rightarrow \frac{r}{2\sin\theta\cos\theta} = \frac{R}{\sin\theta}$$

$$\boxed{r = 2R\cos\theta}$$

Q. A particle is moving under a central force if eq^n of path of the particle is $r = Ae^{K\theta}$ What is Potential Energy of the particle.

Sol^n

$r = Ae^{K\theta}$ is eq^n of path becoz this relation b/w two co-ordinate and also it is time independent. when it is written in differential form then it is KLa differential eq^n of path.

Ans:- $f \propto \frac{1}{r^3}$

$$V(r) = \frac{-(K^2+1)l^2}{2mr^2}$$

T.E. = 0

For total energy:-

T.E. = K.E. + P.E.

$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + P.E.$

$r = Ae^{K\theta}$

$\dot{r} = KAe^{K\theta}\dot{\theta}$

$\dot{r} = Kr\dot{\theta}$

$\dot{\theta} = \frac{l}{mr^2}$

* Effective Potential :- [Potential Energy] :-

It is introduce to convert 2-D problem (in central force motion) in to 1-D problem.

Total Energy -

$$E = K.E + P.E. \Rightarrow E = \frac{1}{2}mv^2 + V(r)$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \underbrace{V(r)}_{\text{Actual P.E.}}$$

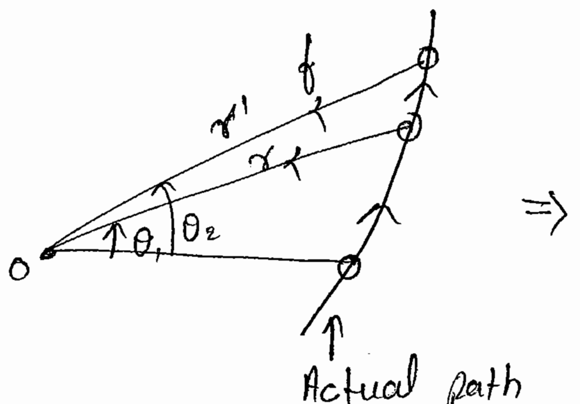
Put $\dot{\theta} = \dfrac{l}{mr^2}$

$$E = \underbrace{\frac{1}{2}m\dot{r}^2}_{K.E.} + \underbrace{\frac{l^2}{2mr^2} + V(r)}_{\text{P.E. or Eff. Pot.}}$$

$$\boxed{E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r)}$$

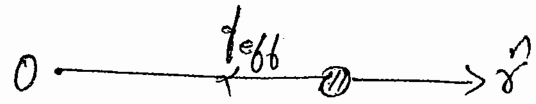
$$\boxed{V_{eff} = V(r) + \frac{l^2}{2mr^2}}$$

$$\boxed{f_{eff} = -\frac{\partial V_{eff}}{\partial r}}$$

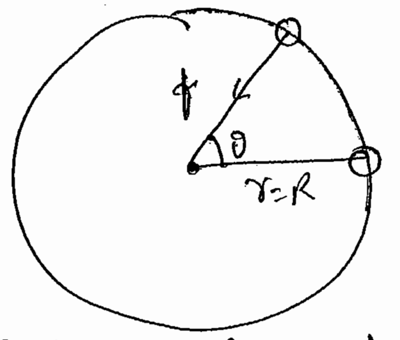


2-D Problem

$\Rightarrow$

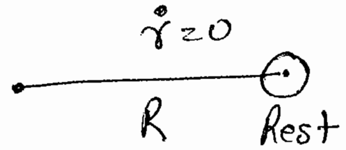


1-D Problem



2-D circular motion

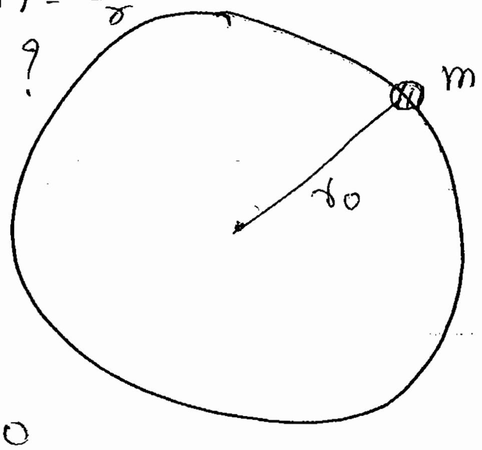
$\Rightarrow$



$$\boxed{f_{eff} = 0}, \quad \boxed{E = V_{eff}.}$$

Q.34

Sol^n It is circular motion



$\therefore$ $f_{eff} = 0$

$$\frac{\partial V_{eff}}{\partial r} = 0$$

$$\frac{\partial}{\partial r}\left(\frac{-K}{r} + \frac{l^2}{2mr^2}\right)_{r=r_0} = 0$$

$$\left[\frac{+K}{r^2} + \left(-\frac{l^2}{mr^3}\right)\right]_{r=r_0} = 0$$

$$\frac{K}{r_0^2} = \frac{l^2}{mr_0^3}$$

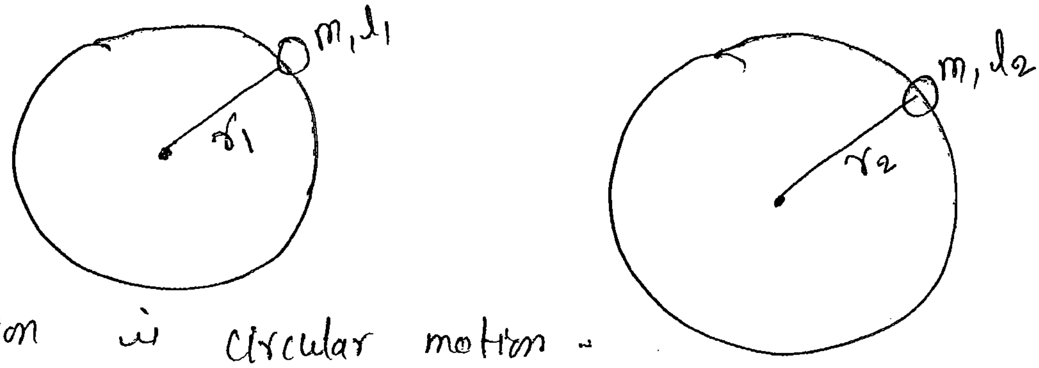
$$l^2 = Kmr_0$$

$$\boxed{l = \sqrt{Kmr_0}}$$ Ans

B.A.5
Q.20

Sol^n Given $\Rightarrow \frac{l_1}{l_2} = 2$ $\qquad V(r) = \frac{1}{2}kr^2$

$$\frac{r_1}{r_2} = ?$$



∵ motion is circular motion.

So $F_{eff} = 0$

$$\frac{\partial V_{eff}}{\partial r} = 0$$

$$\frac{\partial}{\partial r}\left(\frac{1}{2}Kr^2 + \frac{l^2}{2mr^2}\right) = 0$$

$$Kr - \frac{l^2}{mr^3} = 0$$

$$l = \sqrt{Kmr^4}$$

$$l = \sqrt{Km}\, r^2$$

$$l_1 = \sqrt{Km}\, r_1^2 \quad \text{—— (i)}$$

$$l_2 = \sqrt{Km}\, r_2^2 \quad \text{—— (ii)}$$

$$\frac{l_1}{l_2} = \frac{\cancel{\sqrt{Km}}\, r_1^2}{\cancel{\sqrt{Km}}\, r_2^2}$$

$$2 = \frac{r_1^2}{r_2^2}$$

So $\boxed{\frac{r_1}{r_2} = \sqrt{2}}$ Ans

Q.3

Soln

$$V(r) = kr^3$$

$$E = K.E. + P.E. \quad \text{--- (I)}$$

But in circular motion -

$$E = V_{eff.} \quad \text{--- (II)}$$

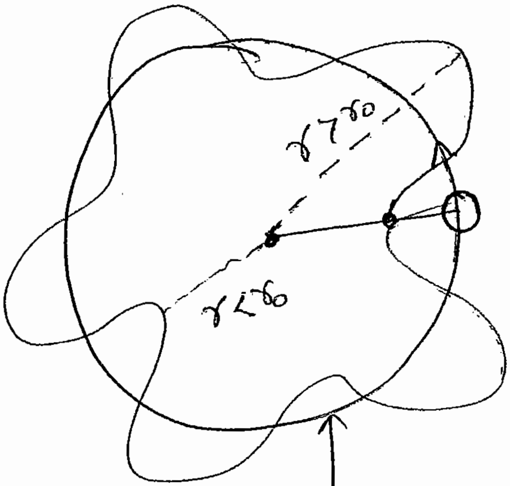
$$K.E. + P.E. = \frac{l^2}{2mr^2} + V(r)$$

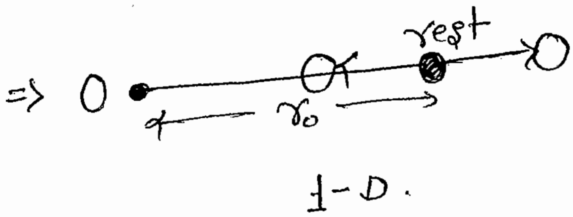
$$K.E. + \cancel{V(r)} = \frac{J^2}{2mr^2} + \cancel{V(r)}$$

$$\boxed{K.E. = \frac{l^2}{2mr^2}} \longrightarrow$$ Always use for circular motion.

* Oscillation about Stable State :-

A stable orbit is circular orbit because ($f_{eff} = 0$)





2-D. Stable circular Orbit

Frequency of Oscillation :-

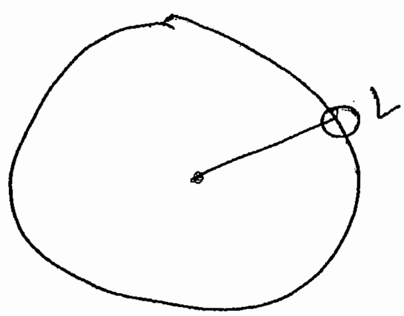
$$\boxed{\omega = \sqrt{\frac{\text{force Constant}}{m}}}$$

When ω = Radial freq. of angular oscillation.

$$\boxed{\text{Force Constant} = \frac{d^2 V_{eff}}{dr^2}\Bigg|_{r=r_0}}$$

where r_0 = radius of stable circular orbit.

8-A-5
Q.82



$\underline{\underline{Sol^n}}$ $V(r) = \frac{-K}{r}$

$$V_{eff} = V(r) + \frac{L^2}{2mr^2}$$

$$V_{eff} = \frac{-K}{r} + \frac{L^2}{2mr^2}$$

Let r_0 be radius of stable circular orbit -

$\therefore$ $f_{eff} = 0$

$$\left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_0} = 0$$

$$\frac{K}{r_0^2} - \frac{L^2}{mr_0^3} = 0$$

$$\boxed{r_0 = \frac{L^2}{mK}}$$

Hence force constant $= \left.\frac{d^2 V_{eff}}{dr^2}\right|_{r=r_0}$

$$= \frac{-2K}{r_0^3} + \frac{3L^2}{mr_0^4}$$

$$= \frac{1}{r_0^3}\left(-2K + \frac{3L^2}{mr_0}\right)$$

$$= \frac{(mk)^3}{L^6}\left(-2K + \frac{3\cancel{L^2} \times \cancel{m}k}{\cancel{m}\cancel{L^2}}\right)$$

$$= \frac{m^3K^3}{L^6}(-2K + 3K) = \frac{m^3k^4}{L^6}$$

Hence angular frequency :-

$$\omega = \sqrt{\frac{\text{force Constant}}{m}}$$

$$= \sqrt{\frac{m^{\cancel{3}} K^4}{L^6 \cancel{m}}} = \frac{mK^2}{L^3}$$

$$\boxed{\omega = \frac{mK^2}{L^3}}$$ Ans

Q. Graph of 'V_{eff} V/s 'r' :-

$$V_{eff} = V(r) + \frac{l^2}{2mr^2}$$

Consider following two limits

$r \to 0$, $r \to \infty$ and draw line in these two limits.

in most of the cases ✓

If $r \to 0$ then higher power in demomenator dominates
If $r \to \infty$ " Lower " " " "

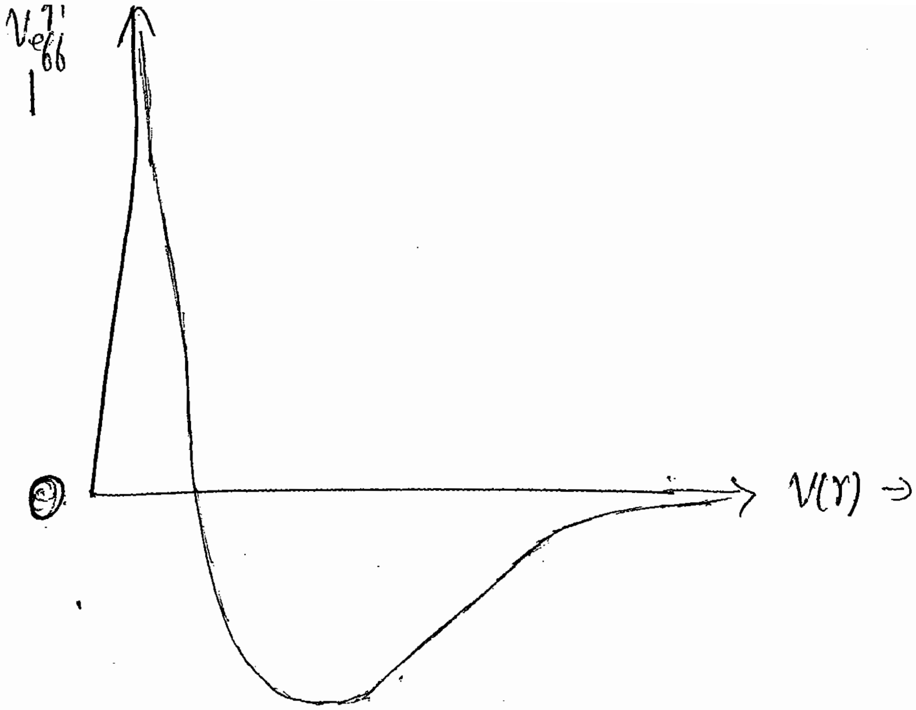
× If $r \to 0$ lowe power " " numerater "
If $r \to \infty$ higher " " " "

Ques Plot V_{eff} for $V(r) = \frac{-K}{r}$

$$V_{eff} = \frac{-K}{r} + \frac{l^2}{2mr^2}$$

$r \to 0$, $V_{eff} \to \infty$

$r \to \infty$, $V_{eff} \to 0$ (from -ve side)



Q: V_{eff} for $V(r) = \frac{-K}{r^3}$

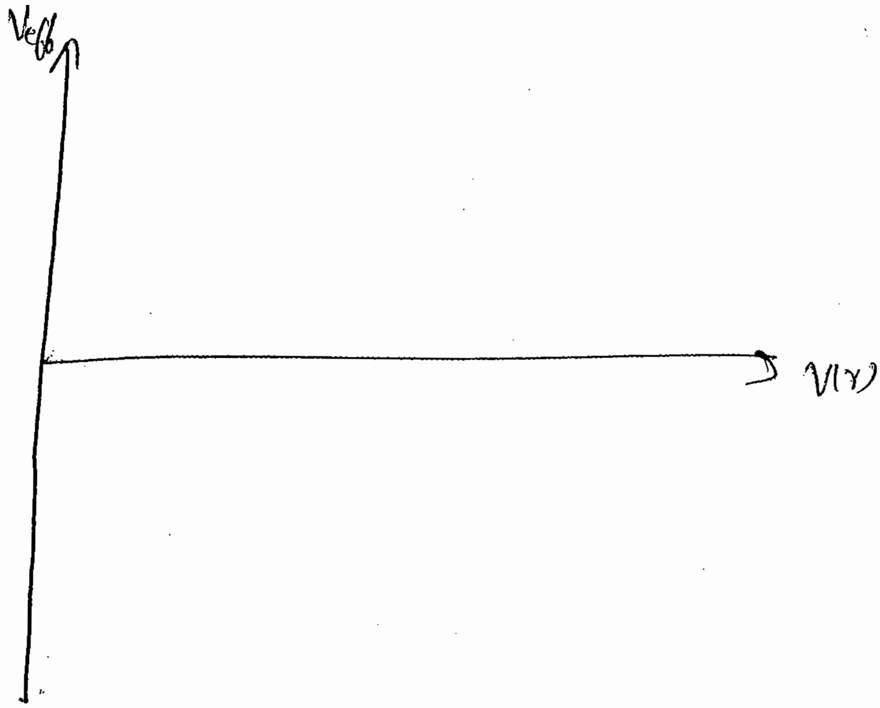
Sol^n

$$V_{eff} = V(r) + \frac{L^2}{2mr^2}$$

$$V_{eff} = -\frac{K}{r^3} + \frac{L^2}{2mr^2}$$

$r \to 0$, $V_{eff} \to -\infty$

$r \to \infty$, $V_{eff} \to 0$ (from +ve side

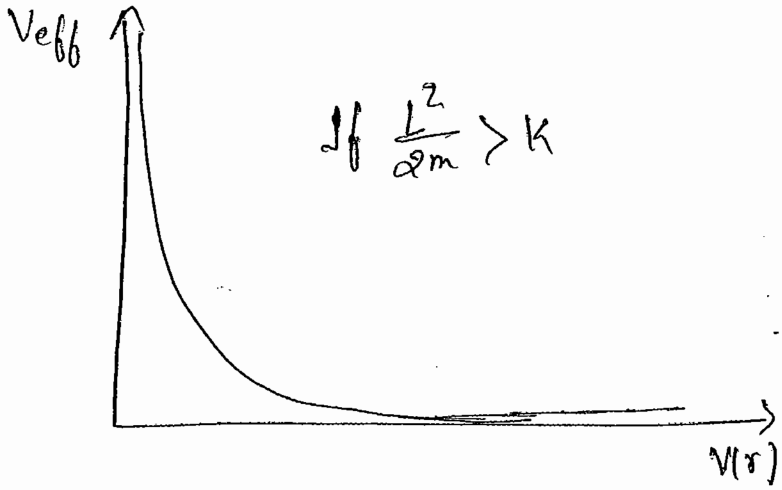


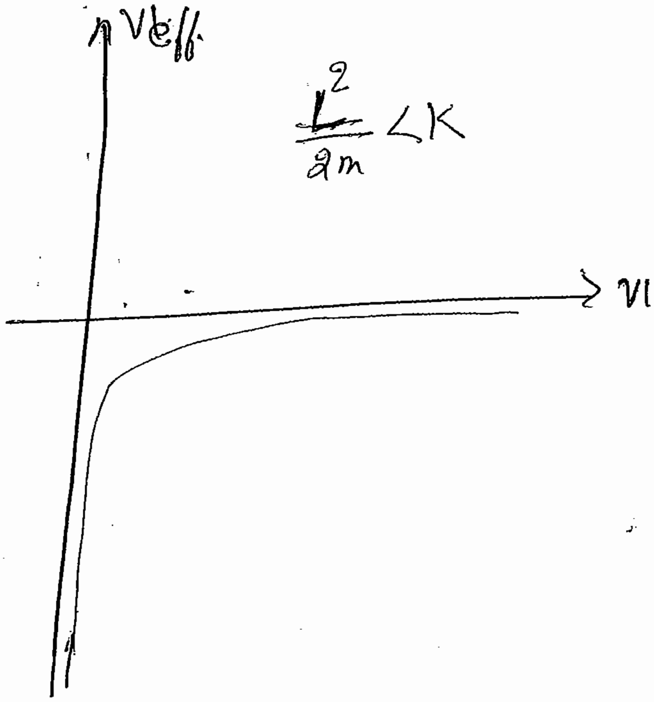
Q. $V(r) = \frac{-K}{r^2}$

$$V_{eff} = -\frac{K}{r^2} + \frac{L^2}{2mr^2}$$

$$= \frac{1}{r^2}\left[\frac{L^2}{2m} - K\right]$$

When $\frac{L^2}{2m} > K$, when $\frac{L^2}{2m} < K$





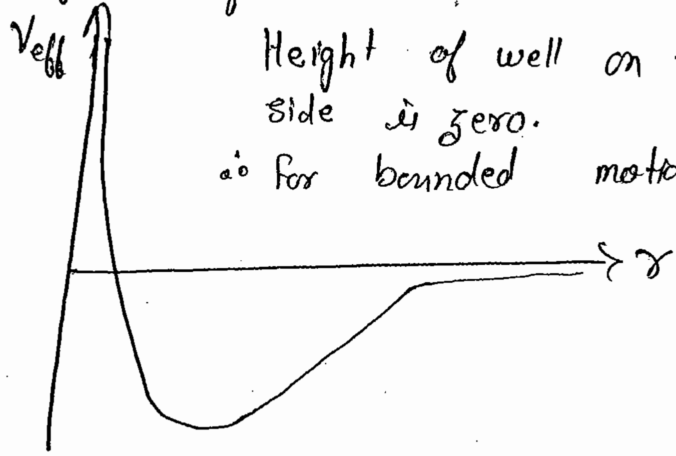
Bounded Motion

If there is a potential well in Veff graph and energy of particle lying inside the well is less than minimum hight of well then motion is bounded.

Example :-

$$V(r) = -\frac{K}{r}$$

$$V_{eff} = \frac{-K}{r} + \frac{L^2}{2mr^2}$$



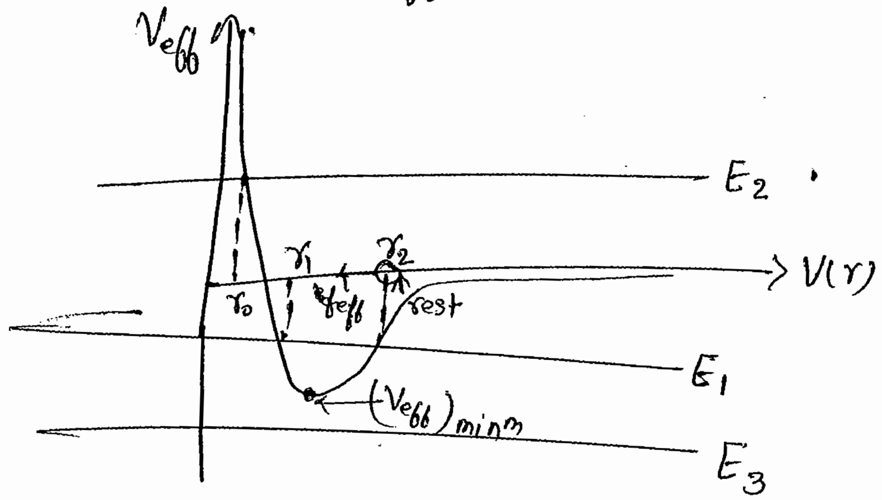
Height of well on right side is zero.
∴ For bounded motion $E < 0$

$$E = \frac{1}{2} m\dot{r}^2 + V_{eff}.$$

$$E - V_{eff} = \frac{1}{2} m\dot{r}^2 \leftarrow \text{+ve or zero.}$$

$$E - V_{eff} \geq 0$$

$E \geq V_{eff} \Rightarrow$ Energy line will be above V_{eff} line.



- E_3 is not allowed.
- For energy E_1 particle moves b/w r_1 and r_2 (these are called terning point).
- for $E = E_2$ particle moves between r_0 and ∞

* Minimum allowed Energy :-

$$\boxed{E_{min} = (V_{eff})_{min}}$$

* Condition for stable orbit :-

$$\boxed{\left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_0} = 0} \quad \text{— (I)}$$

$$\boxed{\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_0} > 0} \quad \text{— (II)}$$

In General :-

If $f = kr^n$

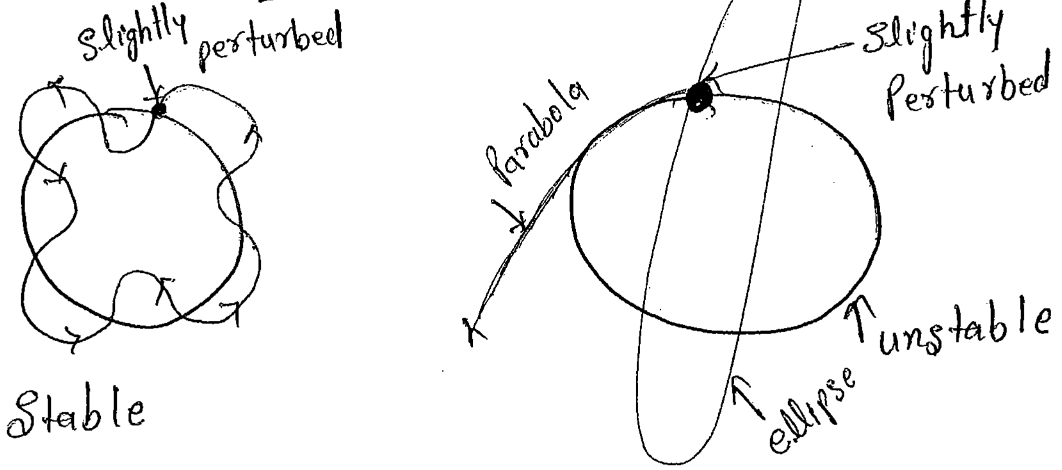
$$V(r) = -\int f(r)\,dr = \frac{-kr^{n+1}}{n+1}$$

$$V_{eff} = V(r) + \frac{l^2}{2mr^2}$$

$$V_{eff.} = -\frac{kr^{n+1}}{n+1} + \frac{l^2}{2mr^2}$$

from the above conditions (1) & (2) for above orbit, we get -

$\boxed{n > -3}$ ← Condition for Stable orbit.



* Condition for Closed Orbit :-

After perturbation slightly the object retrace it's path after some turns. (Ultimately path should closed.).

If $f = kr^n$ then closed orbit is possible only for $n = 1$ and $n = -2$.

[Forces should be attractive]

(I) Hook's Law

(II) Gravitational law. }

Q-5

Sol^n

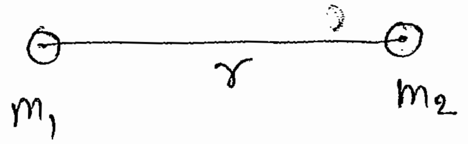
$$f = \frac{K}{r^{3-n}} = K r^{n-3}$$

$n - 3 = 1 \Rightarrow n = 4$

$n = 3 = -2 \Rightarrow n = 1$

$\boxed{n = 4,\ 1}$ Ans

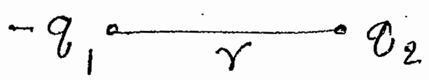
(*) Polar Equation of orbit under $V(r) = \frac{-K}{r}$



$$V(r) = \frac{-G m_1 m_2}{r} = \frac{-K}{r} \qquad K = G m_1 m_2$$

$$F = \frac{-G m_1 m_2}{r^2}$$

$$\boxed{V(r) = -\int f(r)\,dr} = \frac{-K}{r}$$



$$V(r) = \frac{-q_1 q_2}{4\pi\epsilon_0 r} = \frac{-K}{r}$$

$$\Rightarrow \boxed{K = \frac{q_1 q_2}{4\pi \epsilon_0}}$$

Differential Equation of orbit:-

$$\frac{d^2u}{d\theta^2} + u = \frac{-mf}{l^2u^2}$$

$$f(r) = -\frac{\partial V}{\partial r} = -\frac{K}{r^2} = -Ku^2$$

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = +\frac{mk}{l^2}}$$

Sol^n of this eq^n gives -

$$\boxed{r = \frac{l^2/mk}{1 \pm \left(\sqrt{1 + \frac{2El^2}{mk^2}}\right)\cos\theta}} \rightarrow \text{Polar equation of orbit}$$

* Polar form of conic section :-

$$r = \frac{A}{1 \pm e\cos\theta}$$

Compare to get -

$$\boxed{e = \sqrt{1 + \frac{2El^2}{mk^2}}} = \text{eccentricity}$$

e depends on E (energy)

$E > 0$, $e > 1$, orbit is hyperbola

$-\frac{mk^2}{2l^2} < E < 0$, $e < 1$, orbit is ellipse.

$E = 0$, $e = 1$, orbit is parabola

$E = -\frac{mk^2}{2l^2}$, $e = 0$, orbit is circle.

only for $V(r) = -\frac{k}{r}$

Q. A particle of mass m is projected from a height $h = R$ with speed $v = \sqrt{\frac{GM}{R}}$ where M is mass of earth, m is mass of particle. What is the path of the particle? in some direc^n?

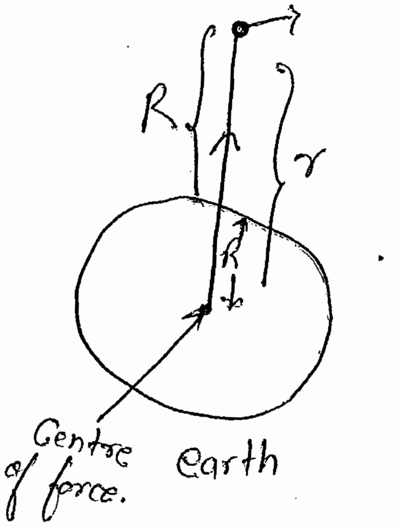
Sol^n "Note

"If a particle is thrown in radial direction it will always be straight line."

$$E = \frac{1}{2} m v^2 + V(r)$$

$$= \frac{1}{2} m \frac{GM}{R} - \frac{K}{r}$$

$$= \cancel{\frac{GMm}{2R}} - \cancel{\frac{GMm}{2R}}$$



$\therefore$ $\boxed{E = 0}$

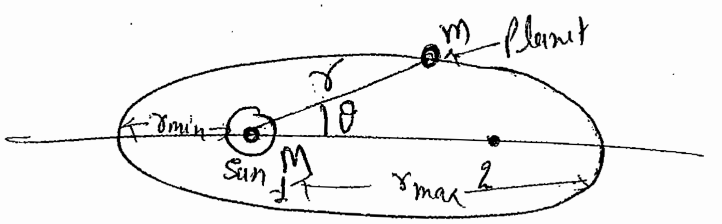
So path is parabola.

* Polar Equation of orbit :-

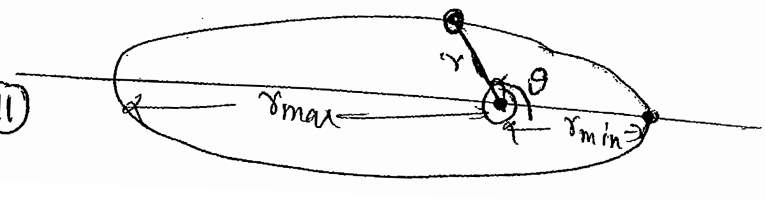
$$\boxed{r = \frac{l^2/mk}{1 \pm e\cos\theta}}$$

Elliptical Orbit :-

$$\boxed{r = \frac{l^2/mk}{1 - e\cos\theta}} \quad \text{—(I)}$$



$$\boxed{r = \frac{l^2/mk}{1 + e\cos\theta}} \quad \text{(II)}$$



r_{max} when $\theta = 0$
r_{min} when $\theta = \pi$ } in eq^n (I)

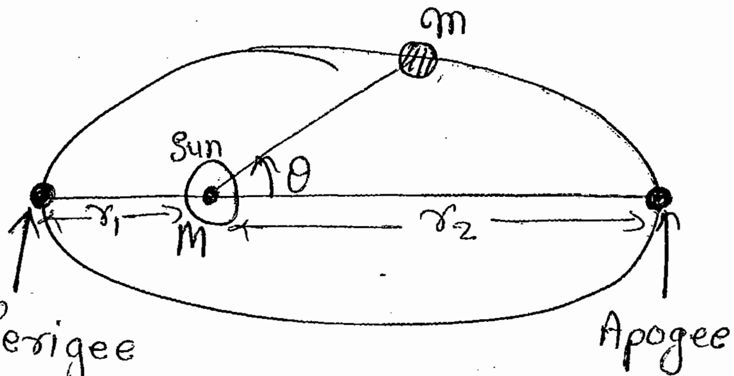
r_{max} when $\theta = \pi$
r_{min} when $\theta = 0$ } in eq^n (II)

* Energy and angular momentum of planet:-

Say:- r_1 and r_2 are given

$$V(r) = -\frac{GMm}{r}$$

Polar equation of orbit-



$$r = \frac{l^2/mk}{1-\left(\sqrt{1+\frac{2El^2}{mk^2}}\right)\cos\theta}$$

$$r_1 = \frac{l^2/mk}{1+\sqrt{1+\frac{2El^2}{mk^2}}} \quad \text{--- (I)}$$

$$r_2 = \frac{l^2/mk}{1-\sqrt{1+\frac{2El^2}{mk^2}}} \quad \text{--- (II)}$$

$$1+\sqrt{1+\frac{2El^2}{mk^2}} = \frac{l^2}{mk}\cdot\frac{1}{r_1} \quad \text{--- (III)}$$

$$1-\sqrt{1+\frac{2El^2}{mk^2}} = \frac{l^2}{mk}\cdot\frac{1}{r_2} \quad \text{--- (IV)}$$

(III) + (IV)

$$2 = \frac{l^2}{mk}\left(\frac{r_1+r_2}{r_1 r_2}\right)$$

$$l = \sqrt{\frac{2r_1 r_2}{(r_1+r_2)}}\, mk$$

$$\boxed{l = m\sqrt{\frac{2GM r_1 r_2}{r_1+r_2}}}$$

Put l in eq^n (1) to get Energy.

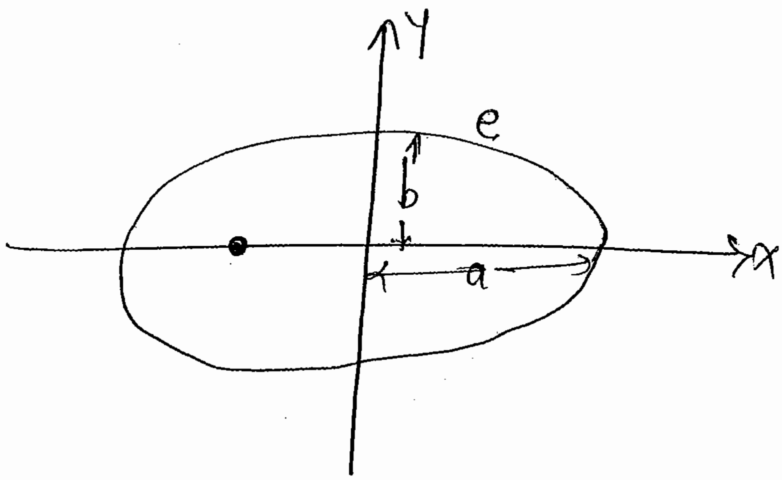
$$\boxed{E = \frac{-GMm}{r_1+r_2}}$$

In terms of ellipse parameter :-

$r_1 = (1-e)a$

$r_2 = (1+e)a$

$e = \sqrt{1-\frac{b^2}{a^2}}$



$$\boxed{r_1+r_2 = 2a}$$

$$\boxed{E = \frac{-GMm}{2a}}$$

$r_1 r_2 = (1-e^2)a^2$

$= \frac{b^2}{a^2}\cdot a^2$

$$\boxed{r_1 r_2 = b^2}$$

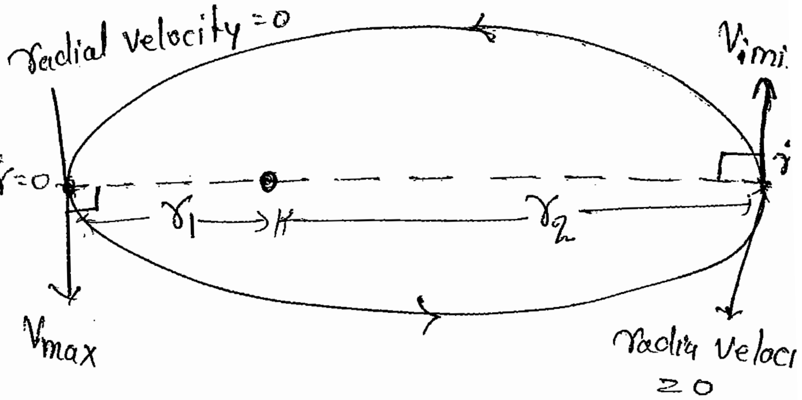
$$L = m\sqrt{\frac{\cancel{2}GMb^2}{\cancel{2}a}}$$

$$\boxed{L = m\sqrt{\frac{GMb^2}{a}}}$$ dependent on a and b.

* Maximum and Minimum Speed in elliptical orbit:-

$$L = m v_{max} r_{min} = m v_{max} r_1$$

$$L = m v_{min} r_{max} = m v_{min} r_2$$



$$L = m\sqrt{\frac{2GMr_1r_2}{(r_1+r_2)}}$$

$$\boxed{v_{min} = \sqrt{\frac{GMr_1}{r_2(r_1+r_2)}}}$$

$$\boxed{v_{max} = \sqrt{\frac{2GMr_2}{r_1(r_1+r_2)}}}$$

$$\boxed{v_{min} = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}}$$

$$\boxed{v_{max} = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}}$$

$r_1 = (1-e)a$

$r_2 = (1+e)a$

$r_1 + r_2 = 2a$

Q. Planet is moving around the sum if ratio of maximum to minimum speed is 2. what is accentricity of orbit?

Soln

$$\frac{v_{max}}{v_{min}} = \frac{\sqrt{\cancel{\frac{GM}{a}}\left(\frac{1+e}{1-e}\right)}}{\sqrt{\cancel{\frac{GM}{a}}\left(\frac{1-e}{1+e}\right)}} = 2$$

$$\frac{1+e}{1-e} = 2$$

$$1+e = 2-2e$$

$$3e = 2-1$$

$$\boxed{e = \frac{1}{3}}$$ Ans

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Sol^n

Polar eq^n of orbit

$$r = \frac{l^2/mk}{1+e\cos\theta}$$

→ Parabola path
⇒ $E=0$, $e=1$
∵ highly elliptical
So we let it is parabola
$\{\cos\theta = \max^m$
$E=0$, $e=1\}$

Ist case :-

$$R_0 = \frac{l^2/mk}{1+|X|}$$

$$R_0 = \frac{l^2}{2mk} \Rightarrow \frac{l^2}{mk} = 2R_0$$

Ist Case :-
Circular orbit :-

$$\because R_0 = \frac{l^2}{2mk} \Rightarrow \frac{l^2}{mk} = 2R_0$$

$$\therefore \frac{V_{max}}{V_{min}} = \frac{1+e}{1-e}$$

$$2 = \frac{1+e}{1-e}$$

Second case :- Circular orbit

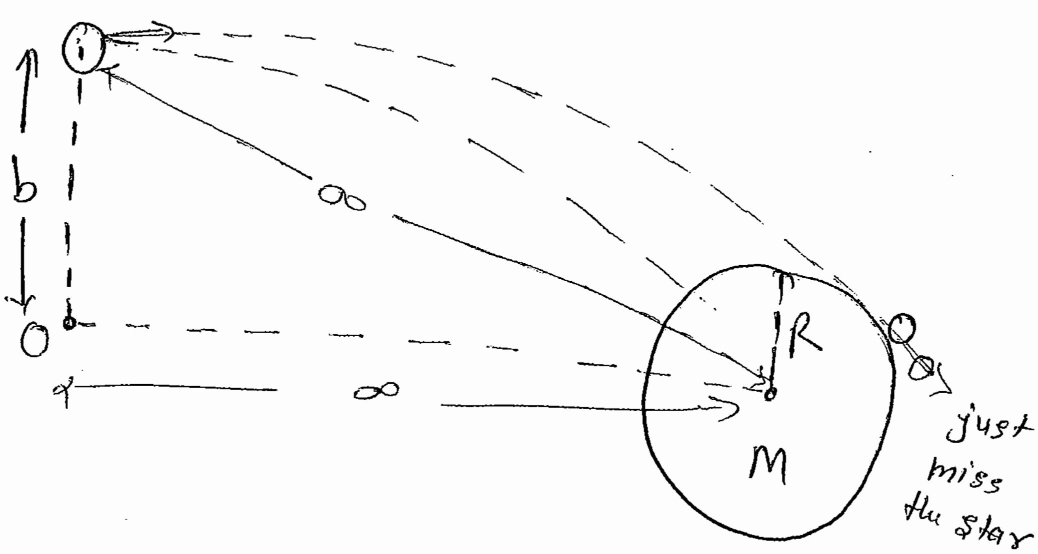
$e=0$

$$R_f = \frac{l^2/mk}{1+0}$$

$$\boxed{R_f = 2R_0}$$ Ans

Q.17

Soln

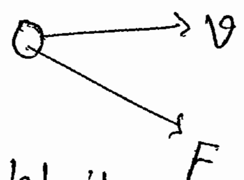


$r_{min} = R$

$$r = \frac{l^2/mk}{1 + e\cos\theta}$$

$$r_{min} = \frac{l^2/mk}{1+e}$$

$$\boxed{R = \frac{l^2/mk}{1+e}}$$



Velocity never cut the direcn of force.

$$R = \frac{l^2/mk}{1 + \sqrt{1 + \frac{2El^2}{mk^2}}}$$

$$E = \frac{1}{2}mv^2 + V(r)$$

$$= \frac{1}{2}mv^2 - \frac{GMm}{\infty} = \frac{1}{2}mv^2$$

$$\boxed{L = mvb}$$

$$\boxed{K = GMm}$$

* Virial Theorem :-

It relates average value of Potential Energy and Kinetic Energy.

If $f = Kr^n$

$$\boxed{\langle K.E.\rangle = \frac{n+1}{2}\langle P.E.\rangle} \Leftarrow \text{Virial Theorem}$$

Ex-

for Gravitation :- $n = -2$

$$\boxed{\langle K.E.\rangle = -\frac{1}{2}\langle P.E.\rangle}$$

for Harmonic Oscillator :-

$$f = -Kr^1$$

$$n = 1$$

$$\boxed{\langle K.E.\rangle = \langle P.E.\rangle}$$

BA-5

Q.7 Potential energy $V = Kr^n$ then Relation b/w K.E. & P.E. is

(a) $\langle T\rangle = \langle V\rangle$ (b) $\langle T\rangle = \frac{n}{2}\langle V\rangle$ (c) $\langle T\rangle = \frac{3}{2}\langle V\rangle$

(d) $\langle T\rangle = 2\langle V\rangle$

Soln -

$$V = Kr^n$$

$$f = -\frac{\partial V}{\partial r} = -Knr^{(n-1)}$$